The Residual Intersection Formula of Type II Exceptional Curves

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1 Preliminary

This paper is a part of the program [Liu1], [Liu3], [Liu4], [Liu5], [Liu6], [Liu7], to understand the family Seiberg-Witten theory and its relationship with the enumeration of nodal or singular curves in linear systems of algebraic surfaces. In [Liu1] a symplectic approach to the universality theorem is given. In [Liu6] the algebraic geometric approach is given. In [Liu7] this result has been interpreted as an enumerative Riemann-Roch formula probing the non-linear information of the linear systems.

The universality theorem implies that for $5n-1$ very ample line bundle $L \mapsto M$, the "number of n-node nodal curves" in a generic n dimensional linear sub-system of $|L|$ can be expressed as a universal polynomial of the characteristic numbers $c_1^2(L), c_1(\mathbf{K}_M)\cdot c_1(L)$, $c_1^2(\mathbf{K}_M)$ and $c_2(M)$, in the spirit of the surface Riemann-Roch formula. On the other hand, for L not sufficiently very ample, the actual virtual number of nodal curves differs from the universal formula predicted by Göttsche [Got]. In [Liu2] the corrections from the type II exceptional classes have been interpreted as a non-linear analogue of second sheaf cohomology.

In this paper, we build up the theory of type II exceptional classes, parallel to the type I theory built up in [Liu1], [Liu5] and [Liu6].

One major application of the type II theory is to define the "virtual number of nodal curves" in $|L|$ on algebraic surfaces without any condition on L.

A direct application of our theory is to argue the vanishing result of type II contributions on universal families of $K3$ surfaces. Once this is achieved, the "virtual numbers of nodal curves" on $K3$ are equal to the polynomials constructed from the universality theorem [Liu6].

Another interesting application of the theory of type II exceptional classes to enumerative problem is the solution of Harvey-Moore conjecture [Liu2] on the enumeration of nodal curves on Calabi-Yau K3 fibrations.

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The layout of the paper is as the following. In section 2, we review the algebraic family Kuranishi models of type II exceptional classes.

Then in section 3, we construct the Kuranishi models explicitly. In section 4, we consider the blowup construction of the algebraic family Seiberg-Witten invariants and prove the main theorem of the paper on the mixed algebraic family Seiberg-Witten invariants attached to a finite collection of type II exceptional classes.

The following is an abbreviated form of our main theorem of the paper, stated in a less technical term. Please refer to theorem 1 on page 28 for the more complete statement.

Main Theorem 1 *Given an algebraic family of algebraic surfaces $\pi : \mathcal{X} \mapsto B$ and a finite collection of type II exceptional*[1] *classes, $e_{II;1}$, $e_{II;2}$, $e_{II;3}$, $\cdots$, $e_{II;p}$ along $\mathcal{X} \mapsto B$ satisfying $e_{II;i} \cdot e_{II;j} \geq 0$ for $i \neq j$, then the localized (excess) contribution of the algebraic family Seiberg-Witten invariant $\mathcal{AFSW}_{\mathcal{X}\mapsto B}(1, C)$ along the locus of co-existence $\times_B^{1\leq i\leq p}\pi_i(\times_B^{1\leq i\leq p}\mathcal{M}_{e_{II;i}})$ of type II exceptional classes is well defined as a mixed algebraic family Seiberg-Witten invariant of $C - \sum_{1\leq i\leq p} e_{II;i}$, manifestly a topological invariant independent of the choices of the family Kuranishi models and the possible deformations of the family $\pi : \mathcal{X} \mapsto B$. As a direct consequence, the residual contribution of the family invariant $\mathcal{AFSW}_{\mathcal{X}\mapsto B}(1, C)$, which is $\mathcal{AFSW}_{\mathcal{X}\mapsto B}(1, C)$ subtracted by the localized excess contribution, is well defined.*

The above theorem also works for the mixed invariants $\mathcal{AFSW}_{\mathcal{X}\mapsto B}(\eta, C)$, $\eta \in \mathcal{A}.(B)$.

The above theorem generalizes the theory of type I exceptional curves developed in [Liu5] and [Liu6] to the cases when the moduli spaces of exceptional curves are not regular.

At the end of the paper, we outline the procedure to extend the scheme to an inductive scheme on hierarchies of finite collections of type II classes. Then we apply the inductive scheme to the universal families of $K3$ surfaces and argue the vanishing results on $K3$ universal families.

2 The Review of Algebraic Family Kuranishi Models of Type II Exceptional Classes

Recall that a type I exceptional class $e_i = E_i - \sum_{j_i} E_{j_i}$ of the fiber bundle of universal spaces $M_{n+1} \mapsto M_n$ has the following two crucial properties:

1. The family moduli space of e_i is smooth of codimension $-\frac{e_i^2 - c_1(\mathbf{K}_{M_{n+1}/M_n})\cdot e_i}{2}$ $= -e_i^2 - 1$ in M_n and can be identified with the closure of an admissible stratum, $\overline{Y(\Gamma_{e_i})}$, for a fan-like admissible graph Γ_{e_i}.

[1]For the definition of exceptional classes, please consult definition 1 on page 3.

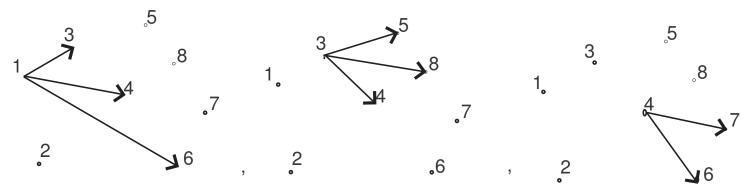

fig.1
A few fan-like admissible graphs with eight vertexes

2. Over each $b \in Y(\Gamma_{e_i})$, the class e_i is represented by a unique holomorphic curve, called type I exceptional curve, in the fiber $M_{n+1}|_b = M_{n+1} \times_{M_n} \{b\}$. Over a Zariski open subset $Y_{\Gamma_{e_i}}$ of $Y(\Gamma_{e_i})$, the curves representing e_i are irreducible. Over a finite union of codimension one loci in $Y(\Gamma_{e_i}) - Y_{\Gamma_{e_i}}$, the curves are disjoint unions of irreducible components, each of them being an irreducible type I exceptional curve.

The fact that the fibrations of type I curves are smooth and "universal" has played a crucial role in understanding the universal nature of the "universality" theorem [Liu1], [Liu6], as a natural extension of surface Riemann-Roch theorem in enumerative geometry.

The above properties about type I exceptional curves are the consequence of the existence of a "canonical" algebraic family Kuranishi model of e_i. These two properties have been used extensively in the proofs of the universality theorem [Liu1], [Liu6]. In this paper, we develop the necessary algebraic technique to deal with type II exceptional curves.

Recall the following definition of exceptional classes which had already appeared in [Liu4], [Liu7],

Definition 1 *A class e is said to be exceptional over $\pi : \mathcal{X} \mapsto B$ if it satisfies the following conditions:*
(i). The fiberwise self-intersection number $\int_B \pi_*(e^2) < 0$.

(ii). The relative degree $deg_{\mathcal{X}/B} e > 0$.

Definition 2 *Consider the universal family $\mathcal{X} = M_{n+1} \mapsto B = M_n$. An exceptional class e is said to be a type II exceptional class if e does not lie in the kernel of $\mathcal{A}.(M_{n+1}) \mapsto \mathcal{A}.(M \times M_n)$.*

For a general (non-universal) fiber bundle $\mathcal{X} \mapsto B$, we also use the term "type II exceptional class" in referring exceptional classes of the fibration $\mathcal{X} \mapsto B$. In

this paper, we often use e_{II} with a subscript II to denote a type II exceptional class.

To simplify our discussion and get to the key point, we impose an additional condition on e_{II}.

Assumption 1 *(Assumption on e_{II}) $deg_{\mathcal{X}/B}(c_1(\mathbf{K}_{\mathcal{X}/B}) - e_{II}) < 0$.*

This implies that $\mathcal{R}^2\pi_*(\mathcal{E}_{e_{II}}) = 0$ by relative Serre duality.

For a type II exceptional class e_{II} of the fiber bundle $\mathcal{X} \mapsto B$, there is usually no canonical choices of the algebraic family Kuranishi models. We know that when a curve representing e_{II} is irreducible, it must be unique in the same fiber. However the reducible representatives may even contain some irreducible component with a non-negative self-intersection. Thus, the following general symptoms have to be kept in mind,

1'. The family moduli space of e_{II}, $\mathcal{M}_{e_{II}} \mapsto B$ (discussed in more detail below), may be not of the expected algebraic family dimension $dim_{\mathbf{C}}B + p_g + \frac{e_{II}^2 - c_1(\mathbf{K})\cdot e_{II}}{2}$. The sub-locus $\subset \mathcal{M}_{e_{II}}$ over which the universal curve representing e_{II} remains irreducible may not be open and dense in $\mathcal{M}_{e_{II}}$ and can be even empty sometimes.

2'. The projection map $\mathcal{M}_{e_{II}} \mapsto B$ is usually not a closed immersion. And for each $b \in B$, the fiber of $\mathcal{M}_{e_{II}} \mapsto B$ above b, $\mathcal{M}_{e_{II}}|_b$, parametrizing all the curves dual to e_{II} in the fiber $\mathcal{X}_b$, can be of positive dimension. This means that there can be (uncountably) many representatives of e_{II} above this given $b \in B$. By the exceptionality condition on e_{II}, this can occur only when the representative contains more than one irreducible component.

Our task is to develop a version of residual intersection formula of the algebraic family Seiberg-Witten invariant based on the general symptoms 1' and 2'. We will demonstrate that there is a well-defined theory of type II exceptional classes parallel to the theory of type I exceptional classes. The basic tool we will use is the construction of the algebraic family Kuranishi models of e_{II}.

2.1 A Short Review About the Kuranishi-Model of e_{II}

In the following, we give a short review about the construction of the algebraic family Kuranishi models of e_{II} and discuss their basic properties. Because our main application is about the universal families, we assume that $\mathcal{R}^2\pi_*\mathcal{O}_{\mathcal{X}}$ is isomorphic to $\mathcal{O}_B^{p_g}$. In this case[2], $febd(e_{II}, \mathcal{X}/B) = p_g$. We will assume implicitly in most of the current paper that $febd(e_{II}, \mathcal{X}/B) = p_g$ to simplify our discussion.

Because in each fiber algebraic surface of the fiber bundle $\mathcal{X} \mapsto B$, the curve (divisor) representing e_{II} may not be unique, we consider the base space,

[2]For the definition of the formal excess base dimension $febd$, please consult definition 4.5. of [Liu3] for more details.

$\mathcal{T}_B(\mathcal{X})$, of relative Pic^0 tori parametrizing all the holomorphic structures of $\mathcal{E}_{e_{II}}$ over B. Let $(\mathbf{V}_{II}, \mathbf{W}_{II}, \Phi_{\mathbf{V}_{II}\mathbf{W}_{II}})$ be an algebraic family Kuranishi model of e_{II}, defined over $\mathcal{T}_B(\mathcal{X})$ and let $\Phi_{\mathcal{V}_{II}\mathcal{W}_{II}} : \mathcal{V}_{II} \mapsto \mathcal{W}_{II}$ be the corresponding morphism of locally free sheaves. Then by its defining property and the simplifying assumption 1 on e_{II} (on page 4), we have $Ker(\Phi_{\mathcal{V}_{II}\mathcal{W}_{II}}) \cong \mathcal{R}^0\pi_*\mathcal{E}_{e_{II}}$ and $Coker(\Phi_{\mathcal{V}_{II}\mathcal{W}_{II}}) \cong \mathcal{R}^1\pi_*\mathcal{E}_{e_{II}}$, where $\mathcal{R}^i\pi_*(\mathcal{E}_{e_{II}})$ over $\mathcal{T}_B(\mathcal{X})$ is the i-th derived image sheaf of $\mathcal{E}_{e_{II}}$ over $\mathcal{X} \times_B \mathcal{T}_B(\mathcal{X})$ along $\pi : \mathcal{X} \times_B \mathcal{T}_B(\mathcal{X}) \mapsto \mathcal{T}_B(\mathcal{X})$.

The kernel $Ker(\Phi_{\mathbf{V}_{II}\mathbf{W}_{II}})$ defines an algebraic sub-cone $\mathbf{C}_{e_{II}}$ over $\mathcal{T}_B(\mathcal{X})$, and its projectification $\mathbf{P}(\mathbf{C}_{e_{II}})$ is nothing but the algebraic family moduli space $\mathcal{M}_{e_{II}}$ of e_{II} over B.

The bundle map $\mathbf{V}_{II} \mapsto \mathbf{W}_{II}$ induces a global section s_{II} of $\mathbf{H}\otimes\pi^*_{\mathbf{P}(\mathbf{V}_{II})}\mathbf{W}_{II}$ over $\mathbf{P}(\mathbf{V}_{II})$ such that $\mathcal{M}_{e_{II}}$ can be identified with the zero locus $Z(s_{II})$ of the section $s_{II} \in \Gamma(\mathbf{P}(\mathbf{V}_{II}), \mathbf{H} \otimes \pi^*_{\mathbf{P}(\mathbf{V}_{II})}\mathbf{W}_{II})$.

As $Z(s_{II})$ may not be regular or be of the expected dimension, we have to rely on intersection theory [F] to construct the virtual fundamental class of $\mathcal{M}_{e_{II}}$.

By applying the concept of localized top Chern class on page 244, section 14.1 of [F],

$$[\mathcal{M}_{e_{II}}]_{vir} \doteq \mathbf{Z}(s_{II}) \in \mathcal{A}_{dim_{\mathbf{C}}\mathcal{T}_B(\mathcal{X})+rank_{\mathbf{C}}\mathbf{V}_{II}-1-rank_{\mathbf{C}}\mathbf{W}_{II}}(\mathcal{M}_{e_{II}})$$

defines a unique cycle class representing the "virtual fundamental class" of the family moduli space $\mathcal{M}_{e_{II}}$ graded by the expected algebraic family Seiberg-Witten dimension [3],

$$ed = dim_{\mathbf{C}}\mathcal{T}_B(\mathcal{X})+rank_{\mathbf{C}}\mathbf{V}_{II}-1-rank_{\mathbf{C}}\mathbf{W}_{II} = dim_{\mathbf{C}}B+q+(p_g-q+\frac{e_{II}^2 - e_{II}\cdot c_1(\mathbf{K}_{\mathcal{X}/B})}{2})$$

$$= dim_{\mathbf{C}}B + p_g + \frac{e_{II}^2 - e_{II}\cdot c_1(\mathbf{K}_{\mathcal{X}/B})}{2}.$$

This virtual fundamental class, i.e. the localized top Chern class $\mathbf{Z}(s_{II})$, can be pushed-forward and mapped into a global cycle class in $\mathcal{A}_{dim_{\mathbf{C}}\mathcal{T}_B(\mathcal{X})+rank_{\mathbf{C}}\mathbf{V}_{II}-1-rank_{\mathbf{C}}\mathbf{W}_{II}}(\mathbf{P}(\mathbf{V}_{II}))$ induced by the inclusion $\mathcal{M}_{e_{II}} \mapsto \mathbf{P}(\mathbf{V}_{II})$.

As we expect, the virtual fundamental class $\mathbf{Z}(s_{II})$ localized in $\mathcal{A}.(\mathcal{M}_{e_{II}})$ is independent to the choices of the algebraic family Kuranishi models of e_{II}.

Proposition 1 *The cycle class $\mathbf{Z}(s_{II}) \in \mathcal{A}.(\mathcal{M}_{e_{II}})$ is independent to the choices of the algebraic family Kuranishi model $(\mathbf{V}_{II}, \mathbf{W}_{II}, \Phi_{\mathbf{V}_{II}\mathbf{W}_{II}})$ of e_{II}.*

Proof: Consider the fiber square

$$\begin{array}{ccc} Z(s_{II}) & \longrightarrow & \mathbf{P}(\mathbf{V}_{II}) \\ \downarrow & & \downarrow s_{II} \\ \mathbf{P}(\mathbf{V}_{II}) & \stackrel{s_{\pi^*_{\mathbf{P}(\mathbf{V}_{II})}\mathbf{W}_{II}\otimes\mathbf{H}}}{\longrightarrow} & \pi^*_{\mathbf{P}(\mathbf{V}_{II})}\mathbf{W}_{II}\otimes\mathbf{H} \end{array}$$

[3]This works for e_{II} with $febd(e_{II}, \mathcal{X}/B) = p_g$.

, where $s_{\pi^*_{\mathbf{P}(\mathbf{V}_{II})}\mathbf{W}_{II}\otimes\mathbf{H}}$ is the zero cross section of $\pi^*_{\mathbf{P}(\mathbf{V}_{II})}\mathbf{W}_{II}\otimes\mathbf{H}$. By the fact that $\mathbf{Z}(s_{II}) = s^!_{\pi^*_{\mathbf{P}(\mathbf{V}_{II})}\mathbf{W}_{II}\otimes\mathbf{H}}[\mathbf{P}(\mathbf{V}_{II})]$, $\mathbf{Z}(s_{II})$ is nothing but the following localized contribution of top Chern class along $Z(s_{II})$,

$$\{c_{total}(\pi^*_{\mathbf{P}(\mathbf{V}_{II})}\mathbf{W}_{II}\otimes\mathbf{H}|_{Z(s_{II})})\cap s_{total}(Z(s_{II}),\mathbf{P}(\mathbf{V}_{II}))\}_{dim_{\mathbf{C}}\mathcal{T}_B(\mathcal{X})+rank_{\mathbf{C}}\mathbf{V}_{II}-1-rank_{\mathbf{C}}\mathbf{W}_{II}}.$$

It suffices to show that the above localized contribution of top Chern class has been independent of the choices of $\mathbf{V}_{II}$ and $\mathbf{W}_{II}$.

Lemma 1 *the localized contribution of top Chern class of $s : \pi^*_{\mathbf{P}(\mathbf{E})}\mathbf{F}\otimes\mathbf{H}$ induced by $\sigma : \mathbf{E}\mapsto\mathbf{F}$ is invariant under the stabilization $\sigma\sim\sigma\oplus id_{\mathbf{G}} : \mathbf{E}\oplus\mathbf{G}\mapsto\mathbf{F}\oplus\mathbf{G}$.*

The lemma is similar to lemma 5.3. in [Liu3].
Proof of the lemma: Under the smooth embedding $\mathbf{P}(\mathbf{E})\hookrightarrow\mathbf{P}(\mathbf{E}\oplus\mathbf{G})$, the normal bundle of $\mathbf{P}(\mathbf{E})$ in $\mathbf{P}(\mathbf{E}\oplus\mathbf{G})$ is isomorphic to the bundle $\pi^*_{\mathbf{P}(\mathbf{E})}\mathbf{G}\otimes\mathbf{H}$, as $\mathbf{P}(\mathbf{E})$ can be viewed as the zero locus of a regular section of $\pi^*_{\mathbf{P}(\mathbf{E}\oplus\mathbf{G})}\mathbf{G}\otimes\mathbf{H}$ induced by the bundle projection $\mathbf{E}\oplus\mathbf{G}\mapsto\mathbf{G}$. So the total Segre class

$$s_{total}(\mathbf{P}(\mathbf{E}),\mathbf{P}(\mathbf{E}\oplus\mathbf{G})) = s_{total}(\pi^*_{\mathbf{P}(\mathbf{E})}\mathbf{G}\otimes\mathbf{H}),$$

and

$$s_{total}(Z(s_{II}),\mathbf{P}(\mathbf{E}\oplus\mathbf{G})) = s_{total}(\mathbf{P}(\mathbf{E}),\mathbf{P}(\mathbf{E}\oplus\mathbf{G}))\cap s_{total}(Z(s_{II}),\mathbf{P}(\mathbf{E}))$$

$$= c_{total}(-\pi^*_{\mathbf{P}(\mathbf{E})}\mathbf{G}\otimes\mathbf{H})\cap s_{total}(Z(s_{II}),\mathbf{P}(\mathbf{E})).$$

Thus

$$c_{total}(\pi^*_{\mathbf{P}(\mathbf{E}\oplus\mathbf{G})}(\mathbf{E}\oplus\mathbf{G})\otimes\mathbf{H}|_{Z(s_{II})})\cap s_{total}(Z(s_{II}),\mathbf{P}(\mathbf{E}\oplus\mathbf{G}))$$
$$= c_{total}(\big(\pi^*_{\mathbf{P}(\mathbf{E})}(\mathbf{E}\oplus\mathbf{G})\otimes\mathbf{H}-\pi^*_{\mathbf{P}(\mathbf{E})}\mathbf{G}\otimes\mathbf{H}\big)|_{Z(s_{II})})\cap s_{total}(Z(s_{II}),\mathbf{P}(\mathbf{E}))$$

$$= c_{total}(\pi^*_{\mathbf{P}(\mathbf{E})}(\mathbf{E}\otimes\mathbf{H}|_{Z(s_{II})})\cap s_{total}(Z(s_{II}),\mathbf{P}(\mathbf{E})).$$

So the localized contribution of top Chern class is invariant under the stabilization. $\Box$

Once the lemma is proved, we may show that the localized top Chern classes defined by any two algebraic family Kuranishi models $(\Phi_{\mathbf{V}_{II}\mathbf{W}_{II}},\mathbf{V}_{II},\mathbf{W}_{II})$ and $(\Phi_{\mathbf{V}'_{II}\mathbf{W}'_{II}},\mathbf{V}'_{II},\mathbf{W}'_{II})$ are equal.

In fact one may stabilize $(\Phi_{\mathbf{V}_{II}\mathbf{W}_{II}},\mathbf{V}_{II},\mathbf{W}_{II})$ into $(\Phi_{\mathbf{V}_{II}\mathbf{W}_{II}}\oplus id_{\mathbf{V}'_{II}},\mathbf{V}_{II}\oplus\mathbf{V}'_{II},\mathbf{W}_{II}\oplus\mathbf{V}'_{II})$ and $(\Phi_{\mathbf{V}'_{II}\mathbf{W}'_{II}},\mathbf{V}'_{II},\mathbf{W}'_{II})$ into $(id_{\mathbf{V}_{II}}\oplus\Phi_{\mathbf{V}'_{II}\mathbf{W}'_{II}},\mathbf{V}_{II}\oplus\mathbf{V}'_{II},\mathbf{V}_{II}\oplus\mathbf{W}'_{II})$, respectively, by applying lemma 1. We find that the localized top Chern classes are stabilized into

$$\{c_{total}(\pi^*_{\mathbf{P}(\mathbf{V}_{II}\oplus \mathbf{V}'_{II})}(\mathbf{W}_{II}\oplus \mathbf{V}'_{II})\otimes \mathbf{H})\cap s_{total}(i_1(\mathcal{M}_{e_{II}}, \mathbf{P}(\mathbf{V}_{II}\oplus \mathbf{V}'_{II}))\}_{ed}$$

and

$$\{c_{total}(\pi^*_{\mathbf{P}(\mathbf{V}_{II}\oplus \mathbf{V}'_{II})}(\mathbf{W}'_{II}\oplus \mathbf{V}_{II})\otimes \mathbf{H})\cap s_{total}(i_2(\mathcal{M}_{e_{II}}, \mathbf{P}(\mathbf{V}_{II}\oplus \mathbf{V}'_{II}))\}_{ed},$$

respectively. Over here $i_1, i_2 : \mathcal{M}_{e_{II}} \hookrightarrow \mathbf{P}(\mathbf{V}_{II}\oplus \mathbf{V}'_{II})$ denote two different imbeddings $\mathcal{M}_{e_{II}} \subset \mathbf{P}(\mathbf{V}_{II}) \subset \mathbf{P}(\mathbf{V}_{II}\oplus \mathbf{V}'_{II})$ and $\mathcal{M}_{e_{II}} \subset \mathbf{P}(\mathbf{V}'_{II}) \subset \mathbf{P}(\mathbf{V}_{II}\oplus \mathbf{V}'_{II})$, respectively.

Firstly, because both $\mathcal{V}_{II}-\mathcal{W}_{II}$ and $\mathcal{V}'_{II}-\mathcal{W}'_{II}$ are equal to $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}(e_{II})\big)-\mathcal{R}^1\pi_*\big(\mathcal{O}_{\mathcal{X}}(e_{II})\big)$ in $K_0(\mathcal{T}_B(\mathcal{X}))$, we have $\mathbf{W}_{II}\oplus \mathbf{V}'_{II} \equiv \mathbf{W}'_{II}\oplus \mathbf{V}_{II}$ and the corresponding total Chern classes are equal.

Secondly, to show that $\rho_1 = s_{total}(i_1(\mathcal{M}_{e_{II}}), \mathbf{P}(\mathbf{V}_{II}\oplus \mathbf{V}'_{II}))$ and $\rho_2 = s_{total}(i_2(\mathcal{M}_{e_{II}}), \mathbf{P}(\mathbf{V}_{II}\oplus \mathbf{V}'_{II}))$ are equal, we notice that i_1, i_2 are within a $\mathbf{P}^1$ pencil of imbeddings of $\mathcal{M}_{e_{II}}$ induced by $j_{a,b} : \mathbf{C}_{e_{II}} \mapsto \mathbf{V}_{II}\oplus \mathbf{V}'_{II}$; $j_{a,b}(v) = aj_1(v)\oplus bj_2(v)$ for $v \in \mathbf{C}_{e_{II}}$, $(a,b) \in \mathbf{C}^2-(0,0)$. Here $j_1 : \mathbf{C}_{e_{II}} \mapsto \mathbf{V}_{II}$ and $j_2 : \mathbf{C}_{e_{II}} \mapsto \mathbf{V}'_{II}$ are the imbeddings of abelian cones projectified into $\mathcal{M}_{e_{II}} \subset \mathbf{P}(\mathbf{V}_{II}), \mathbf{P}(\mathbf{V}'_{II})$, respectively.

So we may consider the embedding $\mathbf{P}^1\times \mathcal{M}_{e_{II}} \mapsto \mathbf{P}^1\times \mathbf{P}(\mathbf{V}_{II}\oplus \mathbf{V}'_{II})$ and the total Segre class $s_{total}(\mathbf{P}^1\times \mathcal{M}_{e_{II}}, \mathbf{P}^1\times \mathbf{P}(\mathbf{V}_{II}\oplus \mathbf{V}'_{II}))$ of its normal cone.

It is clear that if we restrict the total Segre class to different $\{t\}\times \mathcal{M}_{e_{II}}$, $t\in \mathbf{P}^1$, the resulting class $\in \mathcal{A}_\cdot(\mathcal{M}_{e_{II}})$ is independent to t. When $t=0$ and $t=\infty$, we recover ρ_1 and ρ_2, respectively. Thus $\rho_1=\rho_2$.

Because the Chern classes and Segre classes are identified, so are the corresponding localized top Chern classes of $\mathcal{M}_{e_{II}}$. $\square$

Because the localized top Chern class is canonically defined, we will denote them by $[\mathcal{M}_{e_{II}}]_{vir}$.

If we push-forward the cycle class $\mathbf{Z}(s_{II}) = [\mathcal{M}_{e_{II}}]_{vir}$ into $\mathbf{P}(\mathbf{V}_{II})$, then by proposition 14.1(a) on page 244 of [F], the image cycle class is equal to the global $c_{top}(\pi^*_{\mathbf{P}(\mathbf{V}_{II})}\mathbf{W}_{II}\otimes \mathbf{H})\cap[\mathbf{P}(\mathbf{V}_{II})]$.

3 The Construction of Algebraic Family Kuranishi Models

In the previous section, we have discussed the independence of $[\mathcal{M}_{e_{II}}]_{vir}$ to the choices of the algebraic family Kuranishi models of e_{II}. In the following, we will first review the construction of the family Kuranishi models. When we perform the blowup/residual intersection theory construction of algebraic family Seiberg-Witten invariants in subsection 4.2, these explicitly constructed algebraic family Kuranishi models will play a crucial role.

As was mentioned earlier, we focus mostly on the $febd(e_{II}, \mathcal{X}/B) = p_g$ case in the following discussion.

3.1 The Construction of family Kuranishi Model of the Type II Exceptional Curves

In this subsection, we review the explicit construction of the algebraic family Kuranishi model of a type II exceptional class e_{II}.

Let $\pi : \mathcal{X} \mapsto B$ be a fiber bundle of algebraic surfaces and let $\mathcal{T}_B(\mathcal{X})$ be the fiber bundle of the relative Pic^0 tori. As usual, let $\mathcal{E}_{e_{II}} \mapsto \mathcal{T}_B(\mathcal{X})$ be the invertible sheaf corresponding to e_{II}.

Let $D \subset \mathcal{X}$ be an ample effective divisor on $\mathcal{X}$ and let n be a sufficiently large integer.

Lemma 2 *Suppose that $|D|$ is chosen to be sufficiently very ample, then the divisor D in $|D|$ can be chosen such that the composition map $D \subset \mathcal{X} \mapsto B$ is of relative dimension one.*

Proof of lemma 2: For all the closed points $b \in B$, consider the $\mathcal{O}_{\mathcal{X}}(D)$-twisted short exact sequence,

$$0 \mapsto \mathcal{I}_{\mathcal{X}_b}(D) \mapsto \mathcal{O}_{\mathcal{X}}(D) \mapsto \mathcal{O}_{\mathcal{X}_b}(D) \mapsto 0.$$

By theorem 1.5. of [Ko], we may replace D by a suitably large multiple such that $\mathcal{R}^i\pi_*\mathcal{I}_{\mathcal{X}_b}(D) = 0$, $\mathcal{R}^i\pi_*\mathcal{O}_{\mathcal{X}_b}(D) = 0$, for $i > 0$.

So the derived exact sequence from the above short exact sequence generates a short exact sequence [4] for each $b \in B$,

$$0 \mapsto H^0(\mathcal{X}, \mathcal{I}_{\mathcal{X}_b}(D)) \otimes k(b) \mapsto H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(D)) \mapsto H^0(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(D)) \mapsto 0.$$

The space $H^0(\mathcal{X}, \mathcal{I}_{\mathcal{X}_b}(D)) \otimes k(b)$ is the subspace of the global sections $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(D))$ which restricts to the trivial section to $\mathcal{X}_b$. When b moves these vector spaces form a vector bundle, denoted by $\mathbf{U}$. Its rank can be calculated to be

$$\chi(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(D)) - \chi(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(D)) \ll dim_{\mathbf{C}}|D| - dim_{\mathbf{C}}B,$$

if D is chosen such that $\chi(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(D)) \gg dim_{\mathbf{C}}B$. If such an inequality has been achieved, then $dim_{\mathbf{C}}|D|$ is much larger than the dimension of the projective space bundle $\mathbf{P}(\mathbf{U})$ over B. By choosing an element of $|D| - Im(\mathbf{P}(\mathbf{U}))$, it gives rise to a cross section which restricts to non-trivial sections to each fiber $\mathcal{X}_b$. So after replacing the original D by the defining divisor D of the chosen section, the newly chosen D intersects each $\mathcal{X}_b$ properly and cuts it down into a curve. The lemma is proved. $\Box$

[4] Here $k(b)$ is the residue field of b.

Once we have such a carefully chosen D, we are ready to construct Kuranishi models of e_{II}.

The following short exact sequence

$$0 \mapsto \mathcal{O}_{\mathcal{X}} \otimes \mathcal{E}_{e_{II}} \mapsto \mathcal{O}_{\mathcal{X}}(nD) \otimes \mathcal{E}_{e_{II}} \mapsto \mathcal{O}_{nD}(nD) \otimes \mathcal{E}_{e_{II}} \mapsto 0$$

implies

$$0 \mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}} \otimes \mathcal{E}_{e_{II}}\big) \mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}(nD) \otimes \mathcal{E}_{e_{II}}\big) \mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_{e_{II}}\big)$$

$$\mapsto \mathcal{R}^1\pi_*\big(\mathcal{O}_{\mathcal{X}} \otimes \mathcal{E}_{e_{II}}\big) \mapsto 0,$$

and $\mathcal{R}^1\pi_*\big(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_{e_{II}}\big) \cong \mathcal{R}^2\pi_*\big(\mathcal{E}_{e_{II}}\big)$, because by relative Serre vanishing theorem $\mathcal{R}^i\pi_*\big(\mathcal{O}_{\mathcal{X}}(nD) \otimes \mathcal{E}_{e_{II}}\big) = 0$ for large enough n.

By our simplifying assumption 1 on e_{II} (on page 4), $\mathcal{R}^2\pi_*\big(\mathcal{E}_{e_{II}}\big) = 0$. So $\mathcal{R}^1\pi_*\big(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_{e_{II}}\big) = 0$ and $\mathcal{R}^0\pi_*\big(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_{e_{II}}\big)$ is locally free. Then we may take $\mathcal{V}_{e_{II}}$ and $\mathcal{W}_{e_{II}}$ to be the locally free sheaves $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}(nD) \otimes \mathcal{E}_{e_{II}}\big)$ and $\mathcal{R}^0\pi_*\big(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_{e_{II}}\big)$, respectively.

Set $\mathbf{V}_{e_{II}}$, $\mathbf{W}_{e_{II}}$ to be the vector bundles associated with the locally free sheaves $\mathcal{V}_{e_{II}} = \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}(nD) \otimes \mathcal{E}_{e_{II}}\big)$ and $\mathcal{W}_{e_{II}} = \mathcal{R}^0\pi_*\big(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_{e_{II}}\big)$, respectively. These bundles depend on the choices of D and n and are not canonical[5].

Then $\Phi_{\mathbf{V}_{e_{II}}\mathbf{W}_{e_{II}}} : \mathbf{V}_{e_{II}} \mapsto \mathbf{W}_{e_{II}}$ defines an algebraic family Kuranishi model of e_{II}, as was described in section 3.1. Under the assumption that $e_{II} - c_1(\mathbf{K}_{\mathcal{X}/B})$ is nef, by Riemann-Roch theorem $rank_{\mathbf{C}}(\mathbf{V}_{e_{II}} - \mathbf{W}_{e_{II}}) = 1 - q + p_g + \frac{e_{II}^2 - c_1(\mathbf{K}_{\mathcal{X}/B})\cdot e_{II}}{2}$.

Remark 1 *It is useful to comprehend the geometric meaning of the above Kuranishi model. By adjoining a very ample nD, the family moduli space of curves $\mathcal{M}_{e_{II}}$ is naturally embedded into the family moduli space of a better behaved class $e_{II} + nD$, which [6] has the nice structure of a projective space bundle over $\mathcal{T}_B\mathcal{X}$. Then the sub-locus $\mathcal{M}_{e_{II}}$ is characterized by the cross section of the obstruction bundle induced by $\Phi_{\mathbf{V}_{e_{II}}\mathbf{W}_{e_{II}}}$, which requries the ray of non-zero sections in $H^0(\mathcal{X}_b, \mathcal{E}_{e_{II}} \otimes \mathcal{O}_{\mathcal{X}}(nD)|_b)$ to vanish along the effective divisor nD to recover a curve representing e_{II}.*

In the above argument, we have not made use of the exceptionality property, i.e. definition 1, on e_{II}. So we may replace e_{II} by any $\underline{C}$ or $\underline{C} - e_{II}$ which satisfies the nef condition on $\underline{C} - e_{II} - c_1(\mathbf{K}_{\mathcal{X}/B})$. As before, we still assume $febd(\underline{C}, \mathcal{X}/B) = febd(\underline{C} - e_{II}, \mathcal{X}/B) = p_g$ to simplify our discussion. The above argument still goes through without modification. We can choose the same effective ample D and a uniformly large n such that the sheaf morphisms

[5] We drop the notational dependence of $\mathbf{V}_{e_{II}}$ and $\mathbf{W}_{e_{II}}$ on D and n to simplify our symbols.

[6] Thanks to the sufficiently very ampleness of D and the large number n.

$\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}(nD)\otimes\mathcal{E}_{\underline{C}}\big) \mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\otimes\mathcal{E}_{\underline{C}}\big)$ and $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}(nD)\otimes\mathcal{E}_{\underline{C}-e_{II}}\big) \mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\otimes\mathcal{E}_{\underline{C}-e_{II}}\big)$ define the algebraic family Kuranishi models of $\underline{C}$ and $\underline{C}-e_{II}$, respectively.

We denote the corresponding Kuranishi-model vector bundles by $\mathbf{V}_{\underline{C}}$, $\mathbf{W}_{\underline{C}}$ and $\mathbf{V}_{\underline{C}-e_{II}}$, $\mathbf{W}_{\underline{C}-e_{II}}$, respectively. In the following, we will fix a pair of D and n and discuss the switching of the family Kuranishi models between $\underline{C}$ and $\underline{C}-e_{II}$.

Remark 2 *We notice that if we formally replace $nD+\underline{C}$ by C, nD by $\mathbf{M}(E)E$ and $\underline{C}$ by $C-\mathbf{M}(E)E$, the above algebraic family Kuranishi model of $\underline{C}$ corresponds formally to the canonical algebraic family Kuranishi model of $C-\mathbf{M}(E)E$, introduced in [Liu3] and used heavily in [Liu6],*

$$\mathcal{R}^0\pi_*\big(\mathcal{O}_{M_{n+1}}\otimes\mathcal{E}_C\big) \mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{M}(E)E}\otimes\mathcal{E}_C\big).$$

This analogue provides us an easy way to memorize and link their family Kuranishi models.

3.2 The Switching of Family Kuranishi Models Involving type II Exceptional Classes

In the following, we compare the Kuranishi datum of $\underline{C}$ and $\underline{C}-e_{II}$ (using only one e_{II}). Consider the pull-back of the Kuranishi models datum of $\underline{C}$ and $\underline{C}-e_{II}$ from B to $\mathcal{M}_{e_{II}}$ by the natural projection $\mathcal{M}_{e_{II}} \mapsto B$.

Let $\mathbf{e}_{II}$, with the following commutative diagram,

$$\begin{array}{ccc} \mathbf{e}_{II} & \hookrightarrow & \mathcal{X}\times_B \mathcal{M}_{e_{II}} \\ \downarrow & & \downarrow \\ \mathcal{M}_{e_{II}} & = & \mathcal{M}_{e_{II}} \end{array}$$

denote the universal type II curve over $\mathcal{M}_{e_{II}}$.

Proposition 2 *Consider the pull-backs of the family Kuranishi models of $\underline{C}$ and $\underline{C}-e_{II}$ to $\mathcal{M}_{e_{II}}$. Between the pull-backs to $\mathcal{M}_{e_{II}}$ of the algebraic family Kuranishi models of $\underline{C}$ and $\underline{C}-e_{II}$ constructed following the recipe in subsection 3.1, there is a commutative diagram of "columns of short exact sequence" of locally free sheaves,*

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}(nD)\otimes\mathcal{E}_{\underline{C}-e_{II}}\big) \arrow[rr, "\Phi_{\mathcal{V}_{\underline{C}-e_{II}}\mathcal{W}_{\underline{C}-e_{II}}}"] \arrow[d] & & \mathcal{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\otimes\mathcal{E}_{\underline{C}-e_{II}}\big) \arrow[d] \\
\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}(nD)\otimes\mathcal{E}_{\underline{C}}\big)\otimes\mathcal{H}_{II} \arrow[rr, "\Phi_{\mathcal{V}_{\underline{C}}\mathcal{W}_{\underline{C}}}\otimes id_{\mathcal{H}_{II}}"] \arrow[d] & & \mathcal{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\otimes\mathcal{E}_{\underline{C}}\big)\otimes\mathcal{H}_{II} \arrow[d] \\
\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}}(nD)\otimes\mathcal{E}_{\underline{C}}\big)\otimes\mathcal{H}_{II} \arrow[rr] \arrow[d] & & \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(nD)\otimes\mathcal{E}_{\underline{C}}\big)\otimes\mathcal{H}_{II} \arrow[d] \\
0 & & 0
\end{tikzcd}$$

Here the hyperplane bundle $\mathcal{H}_{II}\mapsto\mathcal{M}_{e_{II}}$ in the above diagram is induced from the embedding $\mathcal{M}_{e_{II}}\hookrightarrow\mathbf{P}(\mathbf{V}_{e_{II}})$.

Sketch of the Proof: Both of the columns are the derived long exact sequences from some twisted versions of the following short exact sequence, pull-back to $\mathcal{X}\times_B\mathcal{M}_{e_{II}}\mapsto\mathcal{M}_{e_{II}}$,

$0\mapsto\mathcal{O}_{\mathcal{X}}(-\mathbf{e}_{II})\mapsto\mathcal{O}_{\mathcal{X}}\mapsto\mathcal{O}_{\mathbf{e}_{II}}\mapsto 0$

and its restriction to the non-reduced nD. These derived sequences are short exact as n has been chosen to be large enough to guarantee the vanishing of $\mathcal{R}^i\pi_*\big(\mathcal{O}_{\mathcal{X}}(nD)\otimes\mathcal{E}\big)$, $i>0$ for either $\mathcal{E}=\mathcal{E}_{\underline{C}}$ or $\mathcal{E}_{\underline{C}-e_{II}}$. The commutativity of the sequences follow from the commutativity of the tensoring operations (by tensoring the defining sections of $\mathbf{e}_{II}\subset\mathcal{X}\times_B\mathcal{M}_{e_{II}}$) and the restriction to nD. The map of the last row is induced by the derived exact sequence of $0\mapsto\mathcal{O}_{\mathbf{e}_{II}}\otimes\mathcal{E}_{\underline{C}}\mapsto\mathcal{O}_{\mathbf{e}_{II}}(nD)\otimes\mathcal{E}_{\underline{C}}\mapsto\mathcal{O}_{\mathbf{e}_{II}\cap nD}(nD)\otimes\mathcal{E}_{\underline{C}}\mapsto 0$.

If $\mathbf{e}_{II}\cap nD\mapsto\mathcal{M}_{e_{II}}$ has been a finite morphism, the sheaf $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(nD)\otimes\mathcal{E}_{\underline{C}}\big)$ is automatically locally free with its rank equal to the relative length of $\mathbf{e}_{II}\cap nD\mapsto\mathcal{M}_{e_{II}}$. Without the finiteness assumption of the morphism $\mathbf{e}_{II}\cap nD\mapsto\mathcal{M}_{e_{II}}$, we check the locally freeness of $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(nD)\otimes\mathcal{E}_{\underline{C}}\big)$ by proving that the second column in the above commutative diagram remains left exact after tensoring with $k(y)$, for all the closed points $y\in\mathcal{M}_{e_{II}}$. $\square$

The exact sequences in proposition 2 imply that $\mathbf{V}_{\underline{C}-e_{II}}$ is a sub-bundle of $\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II}$. Denote the quotient bundles associated with $\mathcal{R}^0\pi_*\big(\mathcal{O}_{e_{II}}(nD)\otimes\mathcal{E}_{\underline{C}}\big)\otimes\mathcal{H}_{II}$ and $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(nD)\otimes\mathcal{E}_{\underline{C}}\big)\otimes\mathcal{H}_{II}$ by $\mathbf{V}'$ and $\mathbf{W}'$, respectively. Then $\mathbf{P}_B(\mathbf{V}_{\underline{C}-e_{II}})$ can be viewed as the smooth sub-scheme of $\mathbf{P}(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II})$ $\cong\mathbf{P}(\mathbf{V}_{\underline{C}})$ defined by the zero locus of the section of $\mathbf{H}\otimes\pi^*_{\mathbf{P}_B(\mathbf{V}_{\underline{C}})}\mathbf{V}'$ induced by $\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II}\mapsto\mathbf{V}'\mapsto 0$.

This implies that we may replace the original ambient space of the family moduli space $\mathcal{M}_{\underline{C}-e_{II}}$ of $\underline{C}-e_{II}$ from $\mathbf{P}_B(\mathbf{V}_{\underline{C}-e_{II}})$ by $\mathbf{P}_B(\mathbf{V}_{\underline{C}})$ and replace the original obstruction bundle $\mathbf{H}\otimes\pi^*_{\mathbf{P}_B(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II})}\mathbf{W}_{\underline{C}-e_{II}}$ by an extended obstruction bundle equivalent to $\mathbf{H}\otimes\pi^*_{\mathbf{P}_B(\mathbf{V}_{\underline{C}\otimes\mathbf{H}_{II}})}(\mathbf{W}_{\underline{C}-e_{II}}\oplus\mathbf{V}')$ in the appropriated K group.

In the following we discuss how the desired "extended obstruction bundle" can be constructed from the standard bundle extension construction.

Consider a bundle extension of $\mathbf{V}'$ by $\mathbf{W}_{\underline{C}-e_{II}}$. All such bundle extensions are classified by the group $Ext^1(\mathbf{V}', \mathbf{W}_{\underline{C}-e_{II}})$. By applying the left exact functor $HOM(\mathbf{V}', \bullet)$ to the short exact sequence $0 \mapsto \mathbf{W}_{\underline{C}-e_{II}} \mapsto \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II} \mapsto \mathbf{W}' \mapsto 0$, we get the following portion of derived long exact sequence,

$$HOM(\mathbf{V}', \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II}) \mapsto HOM(\mathbf{V}', \mathbf{W}') \xrightarrow{\delta} Ext^1(\mathbf{V}', \mathbf{W}_{\underline{C}-e_{II}}) \cdots .$$

Apparently the bundle map $\mathbf{V}' \mapsto \mathbf{W}'$ induced from the sheaf commutative diagram in proposition 2 gives an element in $HOM(\mathbf{V}', \mathbf{W}')$. Its image in $Ext^1(\mathbf{V}', \mathbf{W}_{\underline{C}-e_{II}})$ under the connecting homomorphism determines a bundle extension and therefore defines the new extended obstruction bundle $\mathbf{W}_{new}$. And we have the following defining short exact sequence

$$0 \mapsto \mathbf{W}_{\underline{C}-e_{II}} \mapsto \mathbf{W}_{new} \mapsto \mathbf{V}' \mapsto 0.$$

To show that there is a canonically induced bundle map $\mathbf{W}_{new} \mapsto \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II}$, we take $HOM(\mathbf{W}_{new}, \bullet)$ on the short exact sequence $0 \mapsto \mathbf{W}_{\underline{C}-e_{II}} \mapsto \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II} \mapsto \mathbf{W}' \mapsto 0$ and get the following portion of derived long exact sequence,

$$\cdots \mapsto HOM(\mathbf{W}_{new}, \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II}) \mapsto HOM(\mathbf{W}_{new}, \mathbf{W}') \mapsto Ext^1(\mathbf{W}_{new}, \mathbf{W}_{\underline{C}-e_{II}}) \mapsto \cdots .$$

The composition $\mathbf{W}_{new} \mapsto \mathbf{V}' \mapsto \mathbf{W}'$ induces an element in $HOM(\mathbf{W}_{new}, \mathbf{W}')$. To show that it is the image of an element in $HOM(\mathbf{W}_{new}, \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II})$, it suffices to show that its image into $Ext^1(\mathbf{W}_{new}, \mathbf{W}_{\underline{C}-e_{II}})$ vanishes.

On the other hand, the derived long exact sequence of the contravariant functor $HOM(\bullet, \mathbf{W}_{\underline{C}-e_{II}})$ upon the defining short exact sequence of $\mathbf{W}_{new}$, $0 \mapsto \mathbf{W}_{\underline{C}-e_{II}} \mapsto \mathbf{W}_{new} \mapsto \mathbf{V}' \mapsto 0$ implies that

$$HOM(\mathbf{W}_{\underline{C}-e_{II}}, \mathbf{W}_{\underline{C}-e_{II}}) \mapsto Ext^1(\mathbf{V}', \mathbf{W}_{\underline{C}-e_{II}}) \mapsto Ext^1(\mathbf{W}_{new}, \mathbf{W}_{\underline{C}-e_{II}}) \cdots .$$

The extension class $\in Ext^1(\mathbf{V}', \mathbf{W}_{\underline{C}-e_{II}})$ defining $\mathbf{W}_{new}$ is the image of $id_{\mathbf{W}_{\underline{C}-e_{II}}} \in HOM(\mathbf{W}_{\underline{C}-e_{II}}, \mathbf{W}_{\underline{C}-e_{II}})$. Therefore by exactness of the above derived sequence its image in $Ext^1(\mathbf{W}_{new}, \mathbf{W}_{\underline{C}-e_{II}})$ must vanish.

The bundle morphism $\in HOM(\mathbf{W}_{new}, \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II})$ restricts to the the bundle injection $\mathbf{W}_{\underline{C}-e_{II}} \mapsto \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II}$ on the sub-bundle $\mathbf{W}_{\underline{C}-e_{II}} \subset \mathbf{W}_{new}$.

To show that $\mathbf{V}_{\underline{C}} \otimes \mathbf{H}_{II} \mapsto \mathbf{V}'$ can be lifted [7] to some $\mathbf{V}_{\underline{C}} \otimes \mathbf{H}_{II} \mapsto \mathbf{W}_{new}$, it suffices to show that its image ι in $Ext^1(\mathbf{V}_{\underline{C}} \otimes \mathbf{H}_{II}, \mathbf{W}_{\underline{C}-e_{II}})$ vanishes. This is ensured by the following derived long exact sequence,

[7] Notice that the lifting may not be unique!

$$\cdots HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II}) \mapsto HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}') \mapsto Ext^1(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}_{\underline{C}-e_{II}})\cdots,$$

the fact that ι is the composite image of the element $\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II} \mapsto \mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II}$ in $HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II})$, and the acyclicity of the above long exact sequence.

The following lemma fixes the unique lifting of $\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II} \mapsto \mathbf{V}'$.

Lemma 3 *Among the possible liftings of $\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II} \mapsto \mathbf{V}'$, there is a unique lifting $\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II} \mapsto \mathbf{W}_{new}$ which makes the following diagram commutative,*

$$\begin{array}{ccc}
\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II} & \longrightarrow & \mathbf{W}_{new} \\
\big\downarrow & & \big\downarrow \\
\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II} & \longrightarrow & \mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II}
\end{array}$$

The above commutative diagram ensures the compatibility between the (extended) algebraic family Kuranishi models of $\underline{C}-e_{II}$ and of $\underline{C}$ above $\mathcal{M}_{e_{II}}$.

Proof of Lemma 3: Start with an arbitrary lifting $\kappa:\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II} \mapsto \mathbf{W}_{new}$. We have the following commutative diagram among HOM groups,

$$\begin{array}{ccccc}
HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}_{\underline{C}-e_{II}}) & \mapsto & HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}_{new}) & \mapsto & HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{V}') \\
\big\downarrow = & & \big\downarrow & & \big\downarrow \\
HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}_{\underline{C}-e_{II}}) & \mapsto & HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II}) & \mapsto & HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}')
\end{array}$$

We consider the difference between $\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II} \mapsto \mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II}$ and the image of κ in $HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II})$ and denote it by ζ. The image of ζ in $HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}')$ vanishes because both the image of κ and $\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II} \mapsto \mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II}$ have the same image $\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II} \mapsto \mathbf{W}'$ in $HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}')$. By exactness of the second row above, there exists an element $\rho\in HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}_{\underline{C}-e_{II}})$ which maps onto ζ. By the commutativity of the diagram, denote the image of the element ρ in $HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}_{new})$ by ρ'. Then we replace κ by $\kappa+\rho'$ and it is rather easy to see that the image of $\kappa+\rho'$ in $HOM(\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II},\mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II})$ is $\mathbf{V}_{\underline{C}}\otimes\mathbf{H}_{II} \mapsto \mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II}$. The commutativity of the original diagram follows. $\Box$

It is vital to understand the bundle map $\mathbf{W}_{new} \mapsto \mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II}$. If the bundle map is injective over $\mathcal{M}_{e_{II}}$, then the restriction of the family moduli space of $\underline{C}-e_{II}$ is isomorphic to the restriction of family moduli space of $\underline{C}$ above $\mathcal{M}_{e_{II}}$. Namely, we have the isomorphism $\mathcal{M}_{\underline{C}-e_{II}}\times_B\mathcal{M}_{e_{II}} \cong \mathcal{M}_{\underline{C}}\times_B\mathcal{M}_{e_{II}}$. This will be formulated as a **special condition** on page 28.

On the other hand, the possible failure of the injectivity of $\mathbf{W}_{new} \mapsto \mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II}$ over $\mathcal{M}_{e_{II}}$ may result in the discrepancy of identifying the fiber products $\mathcal{M}_{\underline{C}-e_{II}}\times_B\mathcal{M}_{e_{II}}$ with $\mathcal{M}_{\underline{C}}\times_B\mathcal{M}_{e_{II}}$.

In the following, we identify the kernel cone of $\mathbf{W}_{new} \mapsto \mathbf{W}_{\underline{C}}\otimes\mathbf{H}_{II}$.

Proposition 3 *Given the unique bundle map lifting $\mathbf{V}_{\underline{C}} \otimes \mathbf{H}_{II} \mapsto \mathbf{W}_{new}$ of $\mathbf{V}_{\underline{C}} \otimes \mathbf{H}_{II} \mapsto \mathbf{V}'$ fixed by lemma 3, the following commutative diagram of vector bundles*

$$\begin{array}{ccc} \mathbf{W}_{new} & \longrightarrow & \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II} \\ \downarrow & & \downarrow \\ \mathbf{V}' & \longrightarrow & \mathbf{W}' \end{array}$$

induces an isomorphism between the kernel cones of the horizontal bundle morphisms.

Proof of proposition 3: The vertical arrows are known to be bundle surjections by our construction. The kernels of both the vertical bundle maps of the columns are isomorphic to $\mathbf{W}_{\underline{C}-e_{II}}$. Let $\mathbf{C}_{new}$ and $\mathbf{C}'$ be the kernel sub-cones of $\mathbf{W}_{new} \mapsto \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II}$ and of $\mathbf{V}' \mapsto \mathbf{W}'$, respectively. We prove that $\mathbf{C}_{new} \mapsto \mathbf{C}'$ induced from $\mathbf{W}_{new} \mapsto \mathbf{V}'$ is an isomorphism of abelian cones. , i.e., it suffices to show that for all the closed $b \in B$ the fibers of the cones are isomorphic under $\mathbf{W}_{new}|_b \mapsto \mathbf{V}'|_b$. Because the restriction of $\mathbf{W}_{new} \mapsto \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II}$ to the sub-bundle $\mathbf{W}_{\underline{C}-e_{II}}$ has been injective, so $\mathbf{C}_{new} \cap \mathbf{W}_{\underline{C}-e_{II}}$ is embedded as the zero section sub-cone of $\mathbf{W}_{new}$. By the above commutative diagram it is clear that $\mathbf{C}_{new} \mapsto \mathbf{C}'$. On the other hand for all vectors $t \in \mathbf{C}'|_b$, an arbitrary lifting of t, $\tilde{t} \in \mathbf{W}_{new}|_b$, may or may not lie in the kernel space above b, $\mathbf{C}_{new}|_b$. But the image of $t \in \mathbf{C}'|_b$ into $\mathbf{W}'|_b$ is zero. So the image of $\tilde{t}$ in $\mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II}$ has to map trivially into $\mathbf{W}'|_b$. So one may find a unique element $w(t)$ in $\mathbf{W}_{\underline{C}-e_{II}}|_b$ which maps onto the image of $\tilde{t}$ in $\mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II}|_b$. Because $\mathbf{W}_{\underline{C}-e_{II}} \subset \mathbf{W}_{new}$, $w(t)$ can be viewed as an element in $\mathbf{W}_{new}|_b$ as well and then $t - w(t)$ will be mapped trivially into $\mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II}|_b$. So $t - w(t) \in \mathbf{C}_{new}|_b$. We have shown that every element $t \in \mathbf{C}'$ can be lifted uniquely into an element in $\mathbf{C}_{new}|_b$ and we establish their bijections for all the closed points $b \in B$. $\Box$

Remark 3 *In the above discussion we have hardly used any special property of e_{II}. Suppose that there are p distinct type II exceptional classes $e_{II;1}$, $e_{II;2}$, $\cdots$, $e_{II;p}$ over $\mathcal{X} \mapsto B$, with $e_{II;i} \cdot e_{II;j} \geq 0$ for $i \neq j$. If we replace $\mathcal{M}_{e_{II}}$ by the co-existence locus* [8] *of the type II classes, $\times_B^{i \leq p} \mathcal{M}_{e_{II;i}}$, and replace the single universal curve $\mathbf{e}_{II}$ by the total sum $\sum_{i \leq p} \mathbf{e}_{II;i}$ of all the universal curves, the above discussion can be generalized easily to the cases involving more than one type II class.*

Remark 4 *The above discussion and remark 3 imply that $\mathbf{W}_{new} \mapsto \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II}$ has been the analogue of $\mathbf{W}^{\circ}_{canon} \mapsto \mathbf{W}_{canon}$ in comparing the family Kuranishi models of $C - \mathbf{M}(E)E - \sum e_{k_i}$ and $C - \mathbf{M}(E)E$ involving type I exceptional classes. The main difference is that in the type II case they are not canonical.*

Recall that in [Liu5], [Liu6], we have discussed the relationship of the canonical algebraic family Kuranishi models of $C - \mathbf{M}(E)E$ and $C - \mathbf{M}(E)E - \sum_{i \leq p} e_{k_i}$ over $Y(\Gamma) \times T(M)$, where $e_{k_i} \cdot (C - \mathbf{M}(E)E) < 0$ for $1 \leq i \leq p$.

[8] It will be defined and discussed in more details in subsection 4.2.

Let $\Phi_{\mathbf{V}^{\circ}_{canon}\mathbf{W}^{\circ}_{canon}} : \mathbf{V}^{\circ}_{canon} \mapsto \mathbf{W}^{\circ}_{canon}$ and $\Phi_{\mathbf{V}_{canon}\mathbf{W}_{canon}} : \mathbf{V}_{canon} \mapsto \mathbf{W}_{canon}$ be the canonical algebraic family Kuranishi models of $C - \mathbf{M}(E)E$ and $C - \mathbf{M}(E)E - \sum e_{k_i}$, respectively. Please consult lemma 6 of [Liu5] for their definitions.

Then we have $\mathbf{V}^{\circ}_{canon} = \mathbf{V}_{canon}$ and the bundle map over $Y(\Gamma) \times T(M)$, $\mathbf{W}^{\circ}_{canon}|_{Y(\Gamma)\times T(M)} \mapsto \mathbf{W}_{canon}|_{Y(\Gamma)\times T(M)}$, parallel to the sheaf sequence below.

$$\mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum \Xi_{k_i}} \otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big) \mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{M}(E)E+\sum \Xi_{k_i}} \otimes \mathcal{E}_C\big) \mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{M}(E)E} \otimes \mathcal{E}_C\big) \mapsto \mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum \Xi_{k_i}} \otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big),$$

for the sum of the type I universal curves $\Xi_{k_i} \mapsto Y(\Gamma)$.

It is not hard to establish the following correspondence,

- $\mathbf{V}^{\circ}_{canon} = \mathbf{V}_{canon}$ corresponds to $\mathbf{V}_{\underline{C}} \otimes \mathbf{H}_{II} = \mathbf{V}_{\underline{C}} \otimes \mathbf{H}_{II}$.
- $\mathbf{W}^{\circ}_{canon} \mapsto \mathbf{W}_{canon}$ corresponds to $\mathbf{W}_{new} \mapsto \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{e_{II}}$.

Remark 5 *For the purpose of defining the virtual fundamental classes of $\mathcal{M}_{\underline{C}}$, it is easy to see that the twisting operation from $\mathbf{V}_{\underline{C}} \mapsto \mathbf{W}_{\underline{C}}$ to $\mathbf{V}_{\underline{C}} \otimes \mathbf{H}_{II} \mapsto \mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II}$ does not affect the virtual fundamental class $[\mathcal{M}_{\underline{C}}]_{vir}$ [9] of $\mathcal{M}_{\underline{C}}$.*

3.3 The Vector Bundle $\mathbf{W}'$ and the Type II Class e_{II}

In the previous subsection, section 3.2, we have defined $\mathbf{W}'$ to be the vector bundle associated with $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(nD) \otimes \mathcal{E}_{\underline{C}}\big) \otimes \mathcal{H}_{II}$ and it is a quotient bundle of $\mathbf{W}_{\underline{C}} \otimes \mathbf{H}_{II}$. In this subsection, we would like to bridge $\mathbf{W}'$ with the virtual fundamental class of the type II class e_{II}. The discussion can be extended to more than one $e_{II;i}$ in a parallel manner, once we substitute e_{II} and $\mathbf{e}_{II}$ by $e_{II;i}$ and $\sum \mathbf{e}_{II;i}$, respectively. The general version will be address in subsection 4.1 later.

Firstly we prove a simple lemma,

Lemma 4 *Let $\mathbf{W}'_{II} \mapsto \mathcal{M}_{e_{II}}$ denote the vector bundle associated with the locally free sheaf $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\mathbf{e}_{II} + nD)\big) \otimes \mathcal{H}_{II}$. Then we have the equality on the ranks $rank_{\mathbf{C}}\mathbf{W}'_{II} = rank_{\mathbf{C}}\mathbf{W}'$, and the total Chern class of $\mathbf{W}'$, $c_{total}(\mathbf{W}')$, can be identified with $c_{total}(\mathbf{W}'_{II}) + \eta$, where η is a polynomial of the "$\mathbf{e}_{II} \mapsto \mathcal{M}_{e_{II}}$" push-forward of monomials the variables $c_1(\mathcal{E}_{\underline{C}-e_{II}})$, e_{II} and nD.*

Proof: To determine the ranks of $\mathbf{W}'_{II}$ and $\mathbf{W}'$, it suffices to calculate them at a closed point $b \in \mathcal{M}_{e_{II}}$.

[9] The line bundle $\mathbf{H}_{II}$ does not show up in the theory of the type I exceptional classes. Since for type I classes we use the canonical algebraic family Kuranishi models of e_{k_i} and $\mathbf{H}_{II}$ is reduced to trivial line bundle over $\mathcal{M}_{e_{k_i}} \cong Y(\Gamma_{e_{k_i}})$.

Because nD is very ample in $\mathcal{X}$, we can replace nD by a linearly equivalent very ample divisor such that it intersects with the curve $\mathbf{e}_{II}|_b$ at a finite number of points. It is easy to see by base-change theorem that the ranks of both $\mathbf{W}'_{II}$ and $\mathbf{W}'$ are equal to $\int_{\mathcal{X}} \mathbf{e}_{II}|_b \cap nD \in \mathbf{N}$. Thus they must be equal.

Because the higher derived images vanish, both $\mathcal{R}^0\pi_*(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\mathbf{e}_{II}+nD)) = \pi_*(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\mathbf{e}_{II}+nD))$ and $\mathcal{R}^0\pi_*(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\underline{C}+nD)) = \pi_*(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\underline{C}+nD))$, and their Chern characters can be computed by Grothendieck-Riemann-Roch formula (See chapter 15. and chapter 18. of [F]).

Since $\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\underline{C}+nD)$ can be constructed from $\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\mathbf{e}_{II}+nD)$ by twisting $\mathcal{E}_{\underline{C}-e_{II}}$, the conclusion follows from Grothendieck-Riemann-Roch formula and the fact that total Chern class can be expressed in terms of the Chern character algebraically. $\square$

Next we consider the following short exact commutative diagram[10],

$$\begin{array}{ccccc}
\mathcal{O}_{\mathcal{X}} & \mapsto & \mathcal{O}_{\mathcal{X}}(nD) & \mapsto & \mathcal{O}_{nD}(nD) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{\mathcal{X}}(\mathbf{e}_{II})\otimes\mathcal{H}_{II} & \mapsto & \mathcal{O}_{\mathcal{X}}(\mathbf{e}_{II}+nD)\otimes\mathcal{H}_{II} & \mapsto & \mathcal{O}_{nD}(\mathbf{e}_{II}+nD)\otimes\mathcal{H}_{II} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{\mathbf{e}_{II}}(\mathbf{e}_{II})\otimes\mathcal{H}_{II} & \mapsto & \mathcal{O}_{\mathbf{e}_{II}}(\mathbf{e}_{II}+nD)\otimes\mathcal{H}_{II} & \mapsto & \mathcal{O}_{\mathbf{e}_{II}\cap nD}(\mathbf{e}_{II}+nD)\otimes\mathcal{H}_{II}
\end{array}$$

This diagram is constructed from the twisted versions of the defining short exact sequences of the form $0 \mapsto \mathcal{O} \mapsto \mathcal{O}(\mathbf{D}) \mapsto \mathcal{O}_{\mathbf{D}}(\mathbf{D}) \mapsto 0$ with $\mathbf{D} = \mathbf{e}_{II}$, $\mathbf{e}_{II}$ and $\mathbf{e}_{II}|_{nD}$ for the columns and $\mathbf{D} = nD$, nD and $nD|_{\mathbf{e}_{II}}$ for the rows, respectively. By pushing these exact sequences forward along (the suitable restriction of) $\pi : \mathcal{M}_{e_{II}} \times_B \mathcal{X} \mapsto \mathcal{M}_{e_{II}}$, we get the following commutative diagram of short exact sequences,

$$\begin{array}{ccccc}
\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}\big) & \mapsto & \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}(nD)\big) & \mapsto & \mathcal{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\big) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}(\mathbf{e}_{II})\big)\otimes\mathcal{H}_{II} & \mapsto & \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}(\mathbf{e}_{II}+nD)\big)\otimes\mathcal{H}_{II} & \mapsto & \mathcal{R}^0\pi_*\big(\mathcal{O}_{nD}(\mathbf{e}_{II}+nD)\big)\otimes\mathcal{H}_{II} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}}(\mathbf{e}_{II})\big)\otimes\mathcal{H}_{II} & \mapsto & \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}}(\mathbf{e}_{II}+nD)\big)\otimes\mathcal{H}_{II} & \mapsto & \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\mathbf{e}_{II}+nD)\big)\otimes\mathcal{H}_{II}
\end{array}$$

As usual when we assume $e_{II} - c_1(\mathbf{K}_{\mathcal{X}/B})$ is nef, the second derived image sheaf $\mathcal{R}^2\pi_*\big(\mathcal{O}_{\mathcal{X}}(\mathbf{e}_{II})\big)$ vanishes and we have the following sheaf surjection,

$$\mathcal{R}^1\pi_*\big(\mathcal{O}_{\mathcal{X}}(\mathbf{e}_{II})\big)\otimes\mathcal{H}_{II} \mapsto \mathcal{R}^1\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}}(\mathbf{e}_{II})\big)\otimes\mathcal{H}_{II} \mapsto \mathcal{R}^2\pi_*\big(\mathcal{O}_{\mathcal{X}}\big) \mapsto 0.$$

And this implies that $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}}(\mathbf{e}_{II})\big)\otimes\mathcal{H}_{II}$ is mapped onto the locally free quotient sheaf $\mathcal{R}^2\pi_*\big(\mathcal{O}_{\mathcal{X}}\big)$ of rank p_g.

On the other hand, we have the isomorphism $\mathcal{R}^1\pi_*\big(\mathcal{O}_{nD}(nD)\big) \cong \mathcal{R}^2\pi_*\big(\mathcal{O}_{\mathcal{X}}\big)$, due to the vanishing of $\mathcal{R}^i\pi_*\big(\mathcal{O}_{\mathcal{X}}(nD)\big)$ for $i > 0$ and $n \gg 0$.

[10] We pull back $\mathcal{X} \mapsto B$ by the mapping $\mathcal{M}_{e_{II}} \mapsto B$ and abbreviate $\mathcal{M}_{e_{II}} \times_B \mathcal{X}$ by the same notation $\mathcal{X}$.

Thus we have the following commutative diagram of sheaves[11],

$$
\begin{array}{ccccccc}
\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}(\mathbf{e}_{II}+nD)\big)\otimes\mathcal{H}_{II} & \mapsto & \mathcal{R}^0\pi_*\big(\mathcal{O}_{nD}(\mathbf{e}_{II}+nD)\big)\otimes\mathcal{H}_{II} & \mapsto & \mathcal{R}^1\pi_*\big(\mathcal{O}_{\mathcal{X}}(\mathbf{e}_{II})\big)\otimes\mathcal{H}_{II} & \mapsto & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}}(\mathbf{e}_{II}+nD)\big)\otimes\mathcal{H}_{II} & \mapsto & \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\mathbf{e}_{II}+nD)\big)\otimes\mathcal{H}_{II} & \mapsto & \mathcal{R}^1\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}}(\mathbf{e}_{II})\big)\otimes\mathcal{H}_{II} & \mapsto & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \mapsto & \mathcal{R}^1\pi_*\big(\mathcal{O}_{nD}(nD)\big) & \overset{\cong}{\mapsto} & \mathcal{R}^2\pi_*\big(\mathcal{O}_{\mathcal{X}}\big) & \mapsto & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

Notice that $\mathcal{R}^0\pi_*\big(\mathcal{O}_{nD}(\mathbf{e}_{II}+nD)\big)$ in the first row is nothing but the $\mathbf{W}_{e_{II}}$ in the datum of algebraic family Kuranishi model of e_{II}. The second row is a part of a four-term exact sequence regarding the fiberwise infinitesimal deformations and obstructions of $\mathbf{e}_{II}$.

The above observation indicates that there exists a $4-$term exact sequence of obstruction vector bundles

$$0\mapsto\mathbf{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\big)\mapsto\mathbf{R}^0\pi_*\big(\mathcal{O}_{nD}(\mathbf{e}_{II}+nD)\big)\otimes\mathbf{H}_{II}\mapsto\mathbf{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\mathbf{e}_{II}+nD)\big)\otimes\mathbf{H}_{II}\mapsto\mathbf{R}^2\pi_*\big(\mathcal{O}_{\mathcal{X}}\big)\mapsto 0,$$

between the algebraic family obstruction bundle $\mathbf{W}_{e_{II}}$ of e_{II} and the vector bundle $\mathbf{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\mathbf{e}_{II}+nD)\big)\otimes\mathbf{H}_{II}$ onto the infinitesimal obstructions $\mathbf{R}^1\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}}(\mathbf{e}_{II})\big)$. Their ranks differ by the geometric genus p_g of the fiber algebraic surfaces of $\mathcal{X}\mapsto B$.

As before let ed denote the expected algebraic family Seiberg-Witten dimension of the type II class e_{II}, $ed=dim_{\mathbf{C}}B+p_g+\frac{e_{II}^2-e_{II}\cdot c_1(\mathbf{K}_{\mathcal{X}/B})}{2}$.

The following proposition shows that the virtual fundamental class of $\mathcal{M}_{e_{II}}$, $[\mathcal{M}_{e_{II}}]_{vir}$, appears naturally within the localized contribution of top Chern class of the bundle $\mathbf{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\big)\otimes\mathbf{H}_{II}\oplus\mathbf{W}'$ along $\mathcal{M}_{e_{II}}$. It will play an essential role in our residual intersection theory approach in section 4.2.

Proposition 4 *Let $\mathbf{W}'_{e_{II}}$ and $\mathbf{W}'$ be the vector bundles associated with the locally free sheaves $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\mathbf{e}_{II}+nD)\big)\otimes\mathcal{H}_{II}$ and $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II}\cap nD}(\underline{C}+nD)\big)\otimes\mathcal{H}_{II}$ over $\mathcal{M}_{e_{II}}$, respectively. Then the localized contribution of top Chern class*

$$\{c_{total}(\mathbf{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\big)\otimes\mathcal{H}_{II}\oplus\mathbf{W}')\cap s(\mathcal{M}_{e_{II}},\mathbf{P}_B(\mathbf{V}_{e_{II}}))\}_{ed-p_g}=[\mathcal{M}_{e_{II}}]_{vir}\cap c_{p_g}(\mathcal{R}^2\pi_*\big(\mathcal{O}_{\mathcal{X}}\big))+\tilde{\eta}.$$

Over here $\tilde{\eta}$ is a cycle class which is a polynomial of the push-forward of monomials in $\underline{C}-e_{II}$, $\mathbf{e}_{II}$ and nD along $\mathbf{e}_{II}\mapsto\mathcal{M}_{e_{II}}$.

Proof of proposition 4: We first recall that $[M_{e_{II}}]_{vir}=\{c_{total}(\mathbf{W}_{e_{II}}\otimes\mathbf{H}_{II}|_{\mathcal{M}_{e_{II}}})\cap s_{total}(\mathcal{M}_{e_{II}},\mathbf{P}(\mathbf{V}_{e_{II}}))\}_{ed}\in\mathcal{A}.(\mathcal{M}_{e_{II}})$, is the localized top Chern class of $\pi^*_{\mathbf{P}(\mathbf{V}_{e_{II}})}\mathbf{W}_{e_{II}}\otimes$

[11] These two diagrams have overlapping blocks.

$\mathbf{H}_{II}$. On the other hand, we have observed from the above discussion that $c_{total}(\mathbf{W}'_{e_{II}})$ is the cap product of $c_{total}(\mathbf{W}_{e_{II}}\otimes \mathbf{H}_{II})$ and $c_{total}(\mathbf{R}^2\pi_*\mathcal{O}_{\mathcal{X}})$.

So by capping with $s_{total}(\mathcal{M}_{e_{II}}, \mathbf{P}(\mathbf{V}_{e_{II}}))$ and by taking the degree $ed-p_g$ term, we find

$$\{c_{total}(\big(\mathbf{R}^0\pi_*\mathcal{O}_{nD}(nD)\otimes \mathbf{H}_{II}\oplus \mathbf{W}'_{II}\big)|_{\mathcal{M}_{e_{II}}})\cap s_{total}(\mathcal{M}_{e_{II}}, \mathbf{P}(\mathbf{V}_{e_{II}}))\}_{ed-p_g}$$

$$=\{c_{total}(\mathbf{W}_{e_{II}}\otimes \mathbf{H}_{II}|_{\mathcal{M}_{e_{II}}})\cap c_{total}(\mathbf{R}^2\pi_*\mathcal{O}_{\mathcal{X}})\cap s_{total}(\mathcal{M}_{e_{II}}, \mathbf{P}(\mathbf{V}_{e_{II}}))\}_{ed-p_g}$$

$$=[\mathcal{M}_{e_{II}}]_{vir}\cap c_{p_g}(\mathbf{R}^2\pi_*\mathcal{O}_{\mathcal{X}}),$$

by using that crucial property that the pairing $\{c_{total}(\mathbf{W}_{e_{II}}\otimes \mathbf{H}_{II}|_{\mathcal{M}_{e_{II}}})\cap s_{total}(\mathcal{M}_{e_{II}}, \mathbf{P}(\mathbf{V}_{e_{II}})\}_{ed-k}=0$, and $c_{p_g+k}(\mathbf{R}^2\pi_*\mathcal{O}_{\mathcal{X}})=0$ for all $k>0$.

Then the equality of our proposition follows from applying lemma 4. $\Box$

Remark 6 *If the formal excess base dimension $febd(e_{II}, \mathcal{X}/B)=0$, then the expected dimension $ed = dim_{\mathbf{C}}B + \frac{e_{II}^2 - e_{II}\cdot c_1(\mathbf{K}_{\mathcal{X}/B})}{2}$. Then the identity in the above proposition should be replaced by*

$$\{c_{total}(\mathbf{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\big)\otimes \mathcal{H}_{II}\oplus \mathbf{W}')\cap s(\mathcal{M}_{e_{II}}, \mathbf{P}_B(\mathbf{V}_{e_{II}}))\}_{ed}=[\mathcal{M}_{e_{II}}]_{vir}+\tilde{\eta}.$$

4 The Blowup Construction of Algebraic Family Seiberg-Witten Invariants

In this section we discuss the blowup construction of the family Seiberg-Witten invariant $\mathcal{AFSW}_{\mathcal{X}\mapsto B}(1, \underline{C})$ with respect to a finite collection of type II exceptional classes $e_{II;1}$, $e_{II;2}$, $\cdots$, $e_{II;p}$, generalizing the blowup and residual intersection theory construction of type I exceptional classes in [Liu6]. One major difference between the theories of type I and type II exceptional classes is that for a type II exceptional class e_{II}, the family moduli space of e_{II}, $\mathcal{M}_{e_{II}}=Z(s_{II})$ may not be regular and the cycle class $[\mathcal{M}_{e_{II}}]_{vir}=\mathbf{Z}(s_{II})$ is typically not equal to $[Z(s_{II})]$. Actually for the canonical algebraic family Kuranishi model of a type I exceptional class e_i we [12] have $\mathbf{V}_{e_i}\cong \mathbf{C}$, the constant line bundle over M_n and $\mathbf{P}(\mathbf{C})\cong M_n$. Moreover, the family moduli space of e_i, the existence locus of e_i over M_n, can be identified with the closure of the admissible stratum $Y(\Gamma_{e_i})$ of the fan-like admissible graph Γ_{e_i} (see the graph on page 2).

As $Y(\Gamma_{e_i})$ is smooth of the expected dimension $dim_{\mathbf{C}}M_n+(e_i^2+1)$, $[Y(\Gamma_{e_i})]$ represents the fundamental class of the family moduli space $\mathcal{M}_{e_i}$.

[12] see section 6.2. of [Liu5].

Let $\mathcal{M}_{e_{II;i}}$ be the family moduli space of $e_{II;i}$ and let $\pi_i : \mathcal{M}_{e_{II;i}} \mapsto B$ be the canonical projection into B. Over the locus $\pi_i(\mathcal{M}_{e_{II;i}}) \subset B$ the class $e_{II;i}$ becomes effective and over their intersection $\cap_{1\leq i\leq p}\pi_i(\mathcal{M}_{e_{II;i}}) \subset B$ all the type II exceptional classes $e_{II;i}$ become effective simultaneously.

Definition 3 *Define $\cap_{1\leq i\leq p}\pi_i(\mathcal{M}_{e_{II;i}}) \subset B$ to be the locus of co-existence of $e_{II;1}$, $e_{II;2}$, $\cdots$, $e_{II;p}$. Define $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}} = \times_B^{1\leq i\leq p}\mathcal{M}_{e_{II;i}}$ to be the moduli space of co-existence of the classes $e_{II;1}$, $\cdots$, $e_{II;p}$.*

Ideally we may expect $\mathcal{M}_{e_{II;i}}$ to be smooth of the expected[13] dimension $dim_{\mathbf{C}}B + p_g + \frac{e_{II}^2 - c_1(\mathbf{K}_{\mathcal{X}/B})\cdot e_{II}}{2}$ and there exists a Zariski open and dense subset of $\mathcal{M}_{e_{II;i}}$, called the "interior" of $\mathcal{M}_{e_{II;i}}$, parametrizing the irreducible curves representing $e_{II;i}$. Then $\mathcal{M}_{e_{II;i}}$ can be viewed as the natural compactification of its open and dense "interior". Under the idealistic assumption, we "expect" that there exists a Zariski-dense open subset of $\times_B^{1\leq i\leq p}\mathcal{M}_{e_{II;i}}$ which parametrizes tuples of irreducible universal type II curves $\mathbf{e}_{II;i}$, $1 \leq i \leq p$.

In the real world the individual $\mathcal{M}_{e_{II;i}}$ may not be smooth, the intersection $\cap_{1\leq i\leq p}\pi_i(\mathcal{M}_{e_{II;i}})$ or the fiber product $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}} = \times_B^{1\leq i\leq p}\mathcal{M}_{e_{II;i}}$ is seldom regular.

The basic philosophy of family Gromov-Taubes theory is to replace the objects $\mathcal{M}_{II;i}$ or $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ by the appropriated virtual fundamental classes and interpret the enumeration of the invariants in term of intersection theory [F]. Thanks to the fact that all $\mathcal{M}_{e_{II;i}}$ are compact, no complicated gluing construction is ever needed.

Because the numerical condition $e_{II;i} \cdot \underline{C} < 0$ we impose on $\underline{C}$ and $e_{II;i}$, any effective representative of $\underline{C}$ over the "interior" of $\cap_{1\leq i\leq p}\pi_i(\mathcal{M}_{e_{II;i}})$ has to break off certain multiples of curves representing $e_{II;i}$, for each $1 \leq i \leq p$. So we may write $\underline{C} = (\underline{C} - \sum e_{II;i}) + \sum e_{II;i}$ formally. Thus we should be able to attach a family invariant of $\underline{C} - \sum_{1\leq i\leq p} e_{II;i}$ to (the virtual fundamental class) of $\times_B^{i\leq p}\mathcal{M}_{e_{II;i}}$ using the geometric information of $e_{II;i}$, $1 \leq i \leq p$ and express the localized contribution un-ambiguously.

In the following, we consider the following general question,

Question: Let $\underline{C}$ be an effective curve class over $\mathcal{X} \mapsto B$ and let $e_{II;1}, e_{II;2}, e_{II;3}, \cdots, e_{II;p}$ be p distinct type II exceptional classes over $\mathcal{X} \mapsto B$ such that $e_{II;i} \cdot \underline{C} < 0$ for all $1 \leq i \leq p$, while $e_{II;i} \cdot e_{II;j} \geq 0$ for $i \neq j$. What is the algebraic family Seiberg-Witten invariant attached to the moduli space of co-existence of $e_{II;i}$, $1 \leq i \leq p$, $\times_B^{i\leq p}\mathcal{M}_{e_{II;i}}$? And what is the residual contribution of the algebraic family invariant of $\underline{C}$ away from this moduli space of co-existence of $e_{II;i}$?

The resolution of the type I analogue of the above question has been the backbone of the proof of "universality theorem" [Liu6].

[13] Assuming $febd(e_{II}, \mathcal{X}/B) = p_g$.

Conceptually the residual contribution of the family invariant represents the contributions to the family invariants from curves in $\underline{C}$ within the family $\mathcal{X} \mapsto B$ which are **NOT** decomposed into a union of curves representing $\underline{C} - \sum_{1\leq i\leq p} e_{II;i}$ and $\sum_{1\leq i\leq p} e_{II;i}$, respectively.

There are a few important guidelines that we impose, based on the type I theory, developed algebraically in [Liu6].

Guideline 1: We require that the localized (excess) contribution of the family invariant of $\underline{C}$ along $\times_B^{1\leq i\leq p}\mathcal{M}_{e_{II;i}}$ to be proportional to the virtual fundamental class of $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}} = \times_B^{1\leq i\leq p}\mathcal{M}_{e_{II;i}}$ and to the virtual fundamental class of $\mathcal{M}_{\underline{C}-\sum_{i\leq p} e_{II;i}}$.

In particular, when either $[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}$ or $[\mathcal{M}_{\underline{C}-\sum_{i\leq p} e_{II;i}}]_{vir}$ vanishes, we require the desired localized contribution to vanish as well.

Guideline 2: Because type II exceptional classes can behave badly in comparison with their type I siblings, the process of identifying the family invariant attached to $\underline{C} - \sum_{1\leq i\leq p} e_{II;i}$ may be more delicate than the theory of type I exceptional classes. But we expect that the resulting family invariant (see theorem 1 for details) can be reduced to the modified algebraic family Seiberg-Witten [14] invariant $\mathcal{AFSW}_{M_{n+1}\times T(M)\mapsto M_n\times T(M)}(c_{total}(\tau_\Gamma), C - \mathbf{M}(E)E - \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$ when $\underline{C} = C - \mathbf{M}(E)E$ and $e_{II;i}$ are reduced[15] to some collections of the type I classes of the universal family $M_{n+1} \mapsto M_n$.

Guideline 3: The localized (excess) contribution of the family invariant of $\underline{C}$ along $\times_B^{i\leq p}\mathcal{M}_{e_{II;i}}$ has to be independent to the algebraic family Kuranishi models chosen for $\underline{C}$, $e_{II;i}$, $1 \leq i \leq p$, etc. In particular, it is independent to $n \gg 0$ and the very ample divisor $D \subset \mathcal{X}$, etc, chosen to define the Kuranishi models.

Guideline 4: The construction we will provide should enable us to generalize to an inductive scheme involving more than one single collection of type II exceptional classes. Because exceptional curves can break up and degenerate within a given family, instead of considering only a single collection of exceptional curves our scheme should work for a whole hierarchy of them.

These few guidelines determine the localized (excess) contribution of the family invariant uniquely, as will be shown in theorem 1.

In the following subsection, some basic knowledge in intersection theory [F] is recalled before we move on to the main theorem of the paper.

[14]Consult definition 13 and 14 of [Liu6] for details.

[15]The type II curves satisfying condition $febd(e_{II;i}, \mathcal{X}/B) = p_g$ have different dimension formulae from the type I curves'. This p_g dimension shift introduces an additional $c_{p_g}(\mathcal{R}^2\pi_*\mathcal{O}_{\mathcal{X}})$ insertion for each type II class.

4.0.1 The Normal Cones and the Fiber Products

Let $X_i \mapsto B$, $1 \leq i \leq n$, be n purely $dim_{\mathbf{C}} X_i$ dimensional schemes over a smooth variety B and let $Y_i \subset X_i$, $1 \leq i \leq n$ be the closed sub-schemes of X_i defined by the zero loci of sections $s_i : X_i \mapsto E_i$ of vector bundles $E_i \mapsto X_i$.

Consider the fiber products $\times_{B;i\leq n} Y_i \subset \times_{B;i\leq n} X_i$. In the sub-section we want to review the intersection product refined on $\times_{B;i\leq n} Y_i$, based on Fulton's general construction [F].

Recall that on page 132, definition 8.1.1. of [F], a refined product $x \cdot_f y$ is defined. Let $f : X \mapsto Y$ with Y non-singular and let $p_X : X' \mapsto X$ and $P_Y : Y' \mapsto Y$ be morphisms of schemes. Let $x \in \mathcal{A}.(X')$, $y \in \mathcal{A}.(Y')$. Then we can define $\gamma_f : X \mapsto X \times Y$ by $\gamma_f(t) = (t, f(t))$ and we have the following commutative diagram,

$$\begin{array}{ccc} X' \times_Y Y' & \mapsto & X' \times Y' \\ \Big\downarrow & & \Big\downarrow \\ X & \stackrel{\gamma_f}{\mapsto} & X \times Y \end{array}$$

Definition 4 *Define* $x \cdot_f y = \gamma_f^!(x \times y)$.

Please consult page 132, proposition 8.1.1. for all the basic properties of $\cdot_f$, including its associativity and commutativity, etc.

As usual, we take $[Y]_{vir} = \mathbf{Z}(s_i)$. Our goal is to determine the virtual fundamental class of $\times_B^{i\leq p}[Y_i]_{vir}$.

Definition 5 *Define* $[\times_B^{1\leq i\leq p} Y_i]_{vir}$ *to be the Gysin pull-back* $\Delta^!(\mathbf{Z}(\oplus s_i))$ *of* $\oplus E_i \mapsto \times X_i$ *by the diagonal morphism* $\Delta : B \mapsto B^p$.

Based on mathematical induction, we may reduce to the $p = 2$ case and prove the following proposition.

Proposition 5 *Let* $Y_1 \subset X_1 \mapsto B$ *and* $Y_2 \subset X_2 \mapsto B$ *be closed sub-schemes over* B *defined as the zero loci of the vector bundles* $E_i \mapsto X_i$. *Then the virtual fundamental class of co-existence* $[\times_B^{i\leq p} Y_i]_{vir}$ *defined in definition 5 is equal to* $[Y_1]_{vir} \cdot_{id_B} [Y_2]_{vir}$.

Proof of proposition 5: Firstly consider the Cartesian product $Y_1 \times Y_2 \subset X_1 \times X_2$. It is clear that $C_{Y_1\times Y_2}(X_1 \times X_2) = C_{Y_1}X_1 \times C_{Y_2}X_2$. The Cartesian products project naturally into $B \times B$ and the fiber product $Y_1 \times_B Y_2$ or $X_1 \times_B X_2$ can be viewed as the pull-back through $\Delta : B \mapsto B \times B$ of $Y_1 \times Y_2 \mapsto B \times B$ or $X_1 \times X_2 \mapsto B \times B$.

The virtual fundamental class of $Y_1 \times Y_2$, $[Y_1 \times Y_2]_{vir}$ is

$$\{c_{total}(E_1 \oplus E_2|_{Y_1\times Y_2}) \cap s_{total}(C_{Y_1\times Y_2}(X_1 \times X_2))\}_{\sum dim_{\mathbf{C}} X_i - \sum rank_{\mathbf{C}} E_i}.$$

We know that $C_{Y_1\times Y_2}(X_1 \times X_2) = C_{Y_1}X_1 \times C_{Y_2}X_2$.

From the following lemma, we can compute its total Segre class.

Lemma 5 *Let* $p_1 : Y_1 \times Y_2 \mapsto Y_1$ *and* $p_2 : Y_1 \times Y_2 \mapsto Y_2$ *be the natural projections. Then the projections induce cones* $p_1^* C_{Y_2} X_2$, $p_2^* C_{Y_1} X_1$ *over* $Y_1 \times Y_2$, *the normal cones of* $Y_1 \times Y_2 \subset Y_1 \times X_2$ *and of* $Y_1 \times Y_2 \subset X_1 \times Y_2$. *We have the following identities on the total Segre classes,*

$$s_{total}(C_{Y_1}X_1 \times C_{Y_2}X_2) = s_{total}(p_2^* C_{Y_1}X_1) \cap s_{total}(p_1^* C_{Y_2}X_2) = s_{total}(C_{Y_1}X_1) \times s_{total}(C_{Y_2}X_2).$$

The lemma is a generalization of the Whitney sum formula of vector bundles to normal cones. For completeness, we offer a simple proof here.

Proof of lemma 5:

If we have either $Y_1 = X_1$ or $Y_2 = X_2$, the above formula is a trivial identity. Let us assume $Y_i \neq X_i$, for $1 \leq i \leq 2$. We blow up X_1, X_2 along Y_1, Y_2, respectively and denote the resulting schemes by $\tilde{X}_1$, $\tilde{X}_2$, respectively. Let D_1, D_2 denote the resulting exceptional divisors.

Then we may blow up $\tilde{X}_1 \times \tilde{X}_2$ along the codimension two $D_1 \times D_2$ and denote the resulting scheme $\tilde{X}_3$ with the exceptional divisor D_3.

On the other hand, we may blow up $X_1 \times X_2$ along $Y_1 \times Y_2$ directly and denote the resulting scheme by $X_1 \widetilde{\times} X_2$ and denote the exceptional divisor by D. Consider the dominated morphism $\tilde{X}_3 \mapsto X_1 \times X_2$, which maps D_3 onto $Y_1 \times Y_2$. By the universal property of scheme theoretical blowing up (proposition II.7.14 of [Ha]), the above map factors through $\tilde{X}_3 \mapsto X_1 \widetilde{\times} X_2$. We have the following commutative diagram,

$$\begin{array}{ccc} \tilde{X}_3 & \mapsto & \tilde{X}_1 \times \tilde{X}_2 \\ \downarrow & & \downarrow \\ X_1 \widetilde{\times} X_2 & \mapsto & X_1 \times X_2 \end{array}$$

By using this commutative diagram, we may push-forward the total Segre class of $C_{D_3}\tilde{X}_3$, $= \sum_{i \geq 0} c_1(\mathcal{O}(-D_3))^i$, to $X_1 \times X_2$ along two paths and the results must match. By using the birational invariance of the total Segre classes under proper birational push-forward (page 74, prop. 4.2 of [F]), we conclude that

$$s_{total}(C_{Y_1 \times Y_2}(X_1 \times X_2)) = s_{total}(p_2^* C_{Y_1}X_1) \cap s_{total}(p_1^* C_{Y_2}X_2).$$

□

By inserting the above identity into the defining equality of $[Y_1 \times Y_2]_{vir}$, we find that $[Y_1 \times Y_2]_{vir} = [Y_1]_{vir} \times [Y_2]_{vir}$.

Moreover, we may take $X = Y = B$ and $f = id_B : B \mapsto B$, $X' = Y_1$ and $Y' = Y_2$, $x = [Y_1]_{vir}$ and $y = [Y_2]_{vir}$ in definition 4. In this case $\gamma_f = \Delta : B \mapsto B \times B$ and the virtual fundamental class of the co-existence locus $[Y_1 \times_B Y_2]_{vir}$ is

$$\Delta^!([Y_1 \times Y_2]_{vir}) = \Delta^!([Y_1]_{vir} \times [Y_2]_{vir}) = [Y_1]_{vir} \times_{id_B} [Y_2]_{vir}.$$

□

Because the refined intersection product $\cdot_f$ is associative, by mathematical induction it is not hard to see that $[\times_B^{i\leq p} Y_i]_{vir} = \cdot_{id_B}^{1\leq i\leq p}[Y_i]_{vir}$.

We have the following simple lemma regarding their push-forwards into the global objects $\in \mathcal{A}.(B)$.

Lemma 6 *Let $p_{Y_i} : Y_i \mapsto B$ denote the proper projection map from Y_i to the smooth base space B. The push-forward image of $[\times_B^{i\leq p} Y_i]_{vir}$ into $\mathcal{A}.(B)$ (with $\cdot$ being the intersection product on B) is the intersection product $p_{Y_1*}[Y_1]\cdot p_{Y_2*}[Y_2]_{vir}\cdots]_{vir}\cdot p_{Y_p*}[Y_p]_{vir} \in \mathcal{A}.(B)$.*

Proof of lemma 6: Because B is non-singular, the intersection product $\cdot$ makes $\mathcal{A}^{\cdot}(B) = \mathcal{A}_{dim_{\mathbf{C}}B-\cdot}(B)$ a commutative, graded ring with unit $[B]$.

On the other hand, there is a commutative diagram,

$$\begin{array}{ccc} \times_B^{i\leq p} Y_i & \longrightarrow & \times_{i\leq p} Y_i \\ \downarrow \times_{i\leq p} p_{Y_i}|_{\times_B^{i\leq p} Y_i} & & \downarrow \times_{i\leq p} p_{Y_i} \\ B & \xrightarrow{\Delta} & B^p \end{array}$$

By theorem 6.2. (a) on page 98 of [F],

$$(\times_{i\leq p} p_{Y_i}|_{\times_B^{i\leq p} Y_i})_* \Delta^! = \Delta^! (\times_{i\leq p} p_{Y_i})_*.$$

Because $(\times_{i\leq p} p_{Y_i})_*(\times_{i\leq p}[Y_i]_{vir}) = \times_{i\leq p} p_{Y_i *}[Y_i]_{vir}$, the lemma follows from example 8.1.9. of [F]. $\Box$

4.1 The Virtual Fundamental Class of $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ and the Extension of Prop. 4

We apply the discussion in subsection 4.0.1 to the concrete situation of algebraic family Kuranishi models of $e_{II;i}$. Let $e_{II;i}, 1 \leq i \leq p$ be p distinct type II exceptional classes over $\mathcal{X} \mapsto B$. As in sub-section 3.1, we can take $\mathcal{V}_{II;i}$ and $\mathcal{W}_{II;i}$ to be $\mathcal{R}^0\pi_*(\mathcal{O}(nD)\otimes \mathcal{E}_{e_{II;i}})$ and $\mathcal{R}^0\pi_*(\mathcal{O}_{nD}(nD)\otimes \mathcal{E}_{e_{II;i}})$, respectively. Then $\Phi_{\mathbf{V}_{II;i}\mathbf{W}_{II;i}} : \mathbf{V}_{II;i} \mapsto \mathbf{W}_{II;i}$ defines the algebraic family Kuranishi model for $e_{II;i}$.

We take $X_i = \mathbf{P}_B(\mathbf{V}_{II;i})$ and $Y_i = \mathcal{M}_{e_{II;i}}$. Then $[Y_i]_{vir}$ has to be defined to be the localized top Chern class of $\pi^*_{\mathbf{P}(\mathbf{V}_{II;i})}\mathbf{W}_{II;i}\otimes \mathbf{H}_{II;i}$, constructed in sub-section 3.1.

By the general discussion in the preceding subsection 4.0.1, prop. 5, we may consider the definition,

Definition 6 *Define the virtual fundamental class $[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}$ of $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}} = \times_B^{1\leq i\leq p}\mathcal{M}_{e_{II;i}}$ to be $\cdot_{id_B}^{1\leq i\leq p}[\mathcal{M}_{e_{II;i}}]_{vir}$.*

The following proposition is the natural extension of proposition 4 to the $p > 1$ case.

Proposition 6 *Let $\mathcal{H}_{II;i}$ be (the restriction of) the hyperplane invertible sheaf of $\mathbf{P}(\mathbf{V}_{II;i})$ to $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$. Let $\mathbf{e}_{II;i} \mapsto \mathcal{M}_{e_{II;i}}$ be the universal curve associated to each $e_{II;i}$ with $febd(e_{II;i}, \mathcal{X}/B) = p_g$. Let $\mathbf{W}'$ be the vector bundle associated to the locally free sheaf $\mathcal{R}^0\pi_*\big(\mathcal{O}_{nD\cap\sum_{i\leq p}\mathbf{e}_{II;i}}(nD+\underline{C})\big)\otimes_{i\leq p}\mathcal{H}_{II;i}$ over $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$. Then the degree $dim_{\mathbf{C}}B + \sum_{1\leq i\leq p}\frac{e^2_{II;i}-c_1(\mathbf{K}_{\mathcal{X}/B})\cdot e_{II;i}}{2}$ term of*

$$\{c_{total}(\oplus_{i\leq p}\mathbf{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\big)\otimes\mathcal{H}_{II;i}\oplus\mathbf{W}'|_{\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}})\cap s_{total}(\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}, \times_B^{1\leq i\leq p}\mathbf{P}(\mathbf{V}_{II;i}))\}$$

can be naturally expanded as

$$[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}\cap c^p_{p_g}(\mathcal{R}^2\pi_*\mathcal{O}_{\mathcal{X}}) + \tilde{\eta}.$$

The class $\tilde{\eta}$ is a polynomial (in terms of $\cdot_{id_B}$) of the push-forwards of algebraic expressions in terms of $\underline{C} - \sum_{i\in J} e_{II;i}$, nD and the various $e_{II;i}$, where the index subset J runs through the subsets of $\{1, 2, \cdots, p\}$.

The proof of this proposition is based on mathematical induction and proposition 4.

Sketch of the Proof: Firstly we notice that each $\mathcal{M}_{e_{II;i}}$ is of expected algebraic family dimension $dim_{\mathbf{C}}B + p_g + \frac{e^2_{II;i}-c_1(\mathbf{K}_{\mathcal{X}/B})\cdot e_{II;i}}{2}$. So $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}} = \times_B^{1\leq i\leq p}\mathcal{M}_{e_{II;i}}$ is of expected dimension $dim_{\mathbf{C}}B + p_g\cdot p + \sum_{i\leq p}\frac{e^2_{II;i}+c_1(\mathbf{K}_{\mathcal{X}/B})\cdot e_{II;i}}{2}$. So the degrees of the formula in the statement of the proposition match. When $p = 1$, the above statement is reduced to proposition 4.

For general p, we consider the Cartesian product $\times^{1\leq i\leq p}\mathbf{P}(\mathbf{V}_{II;i})$ and view the fiber product $\times_B^{1\leq i\leq p}\mathbf{P}(\mathbf{V}_{II;i})$ as its pull-back by the diagonal morphism $\Delta : B \mapsto B^p$.

Consider the following sheaf short exact sequence,

$$\mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{i\leq p-1}\mathbf{e}_{II;i}\cap nD}(\underline{C}-e_{II;p}+nD)\big)\otimes^{i\leq p-1}\mathcal{H}_{II;i} \mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\mathbf{e}_{II;i}\cap nD}(\underline{C}+nD)\big)\otimes^{i\leq p}\mathcal{H}_{II;i}$$

$$\mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{e}_{II;p}\cap nD}(\underline{C}+nD)\big)\otimes^{i\leq p}\mathcal{H}_{II;i}.$$

By our induction hypothesis for $p-1$, applying to the class $\underline{C}' = \underline{C} - e_{II;p}$ and $p-1$ distinct exceptional classes $e_{II;1}$, $e_{II;2}$, $\cdots$, $e_{II;p-1}$, and the $p = 1$ case (proposition 4), applying to $\underline{C}' = \underline{C}$ and $e_{II;p}$, using the computation of lemma 5 we can write the localized top Chern class as

$$\big([\mathcal{M}_{e_{II;1},\cdots,e_{II;p-1}}]_{vir}\cap c^{p-1}_{p_g}(\mathcal{R}^2\pi_*\mathcal{O}_{\mathcal{X}})+\tilde{\eta}_1\big)\cdot_{id_B}\big([\mathcal{M}_{e_{II;p}}]_{vir}\cap c_{p_g}(\mathcal{R}^2\pi_*\mathcal{O}_{\mathcal{X}})+\tilde{\eta}_2\big),$$

where $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are $\cdot_{id_B}$-polynomial expressions of the push-forwards of $\underline{C} - e_{II;p} - \sum_{i\in I'} e_{II;i}$, $I' \subset \{1, \cdots, p-1\}$ and $\underline{C} - e_{II;p}$ and nD, $e_{II;i}$, etc. By a simple calculation, the conclusion follows. □

Remark 7 *When $febd(e_{II;i}, \mathcal{X}/B) = 0$, the term $c^p_{p_g}(\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathcal{X}}\big))$ drops from the above formula in proposition 6.*

4.2 The Stabilization of The Kuranishi Model of $\underline{C}$

Let $(\Phi_{\mathbf{V}_{\underline{C}}\mathbf{W}_{\underline{C}}}, \mathbf{V}_{\underline{C}}, \mathbf{W}_{\underline{C}})$ be the algebraic family Kuranishi model of $\underline{C}$ defined by adopting some nD with $n \gg 0$ and let $(\Phi_{\mathbf{V}_{II;i}\mathbf{W}_{II;i}}, \mathbf{V}_{II;i}, \mathbf{W}_{II;i})$ be the algebraic family Kuranishi models of $e_{II;i}$, $1 \leq i \leq p$, constructed following the recipe of subsection 3.1.

Because the family moduli spaces $\mathcal{M}_{e_{II;i}}$ of the type II classes are embedded in $\mathbf{P}(\mathbf{V}_{II;i})$ and not in B, we have to pull-back the algebraic family Kuranishi model of $\underline{C}$ from B first.

Lemma 7 *Let $\mathbf{E}$ be a vector bundle over $\mathcal{T}_B(\mathcal{X})$, then $\mathcal{T}_B(\mathcal{X})$ can be identified with the subset of $\mathbf{P}(\mathbf{E}\oplus\mathbf{C})$ through the embedding induced by the bundle injection $\mathbf{C} \mapsto \mathbf{E}\oplus\mathbf{C}$, and is the zero locus of the canonical section s of $\pi^*_{\mathbf{P}(\mathbf{E}\oplus\mathbf{C})}\mathbf{E}\otimes\mathbf{H}$ induced by $\mathbf{E}\oplus\mathbf{C} \mapsto \mathbf{E}$.*

Proof of lemma 7: The cross-section $\sigma : \mathcal{T}_B(\mathcal{X}) \mapsto \mathbf{P}(\mathbf{E}\oplus\mathbf{C})$ induced by projectifying $\mathbf{C} \mapsto \mathbf{E}\oplus\mathbf{C}$ is clearly isomorphic to $\mathcal{T}_B(\mathcal{X})$. On the other hand, the kernel of the bundle projection $\mathbf{E}\oplus\mathbf{C} \mapsto \mathbf{E}$ is exactly the trivial sub-bundle $\mathbf{C}$. This implies that $\mathbf{H}^* \mapsto \pi^*_{\mathbf{P}(\mathbf{E}\oplus\mathbf{C})}\mathbf{E}$ induced by $\mathbf{E}\oplus\mathbf{C} \mapsto \mathbf{E}$ is injective exactly off $\sigma(\mathcal{T}_B(\mathcal{X})) \subset \mathbf{P}(\mathbf{E}\oplus\mathbf{C})$. So the canonical section s of $\pi^*_{\mathbf{P}(\mathbf{E}\oplus\mathbf{C})}\mathbf{E}\otimes\mathbf{H}$ vanishes exactly on $\sigma(\mathcal{T}_B\mathcal{X})$ and it is easy to see that s is a regular section of $\pi^*_{\mathbf{P}(\mathbf{E}\oplus\mathbf{C})}\mathbf{E}\otimes\mathbf{H}$. $\square$

Remark 8 *The cycle class of the zero locus $\sigma(\mathcal{T}_B(\mathcal{X}))$, $[\sigma(\mathcal{T}_B(\mathcal{X}))] \in \mathcal{A}.(\mathbf{P}(\mathbf{E}\oplus\mathbf{C}))$ is equal to $c_{top}(\pi^*_{\mathbf{P}(\mathbf{E}\oplus\mathbf{C})}\mathbf{E}\otimes\mathbf{H}) \cap [\mathbf{P}(\mathbf{E}\oplus\mathbf{C})]$.*

Definition 7 *Denote the fundamental cycle class of the zero cross-section $B \mapsto \times_B^p\mathcal{T}_B(\mathcal{X})$ to be $[B]_p$.*

The normal bundle of $[B]_p$ in $\times_B^p\mathcal{T}_B(\mathcal{X})$ is isomorphic to $\mathbf{R}^1\pi_*\mathcal{O}_{\mathcal{X}}^{\oplus p}$.

We replace B by the auxiliary space $B' = \times_B^{1\leq i\leq p}\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})$ and view the original $\times_B^p\mathcal{T}_B(\mathcal{X}) \subset B' = \times_B^{1\leq i\leq p}\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})$ as the regular zero locus of the canonical section of the auxiliary obstruction bundle $\oplus_{1\leq i\leq p}\pi^*_{\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})}\mathbf{V}_{II;i}\otimes\mathbf{H}_{II;i}$.

Then by remark 8 and definition 7 we have to compensate by inserting both the top Chern class $c_{top}(\oplus_{1\leq i\leq p}\pi^*_{\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})}\mathbf{V}_{II;i}\otimes\mathbf{H}_{II;i})$ and the $[B]_p$ into the intersection pairing of the family invariant of $\underline{C}$.

Correspondingly for these p distinct type II classes $e_{II;i}$, we stabilize their algebraic family Kuranishi models by the trivial line bundle $\mathbf{C}$ and get $(\Phi_{\mathbf{V}_{II;i}\mathbf{W}_{II;i}}\oplus id_{\mathbf{C}}, \mathbf{V}_{II;i}\oplus\mathbf{C}, \mathbf{W}_{II;i}\oplus\mathbf{C})$.

By lemma 1, these models are invariant under stabilizations. Then the push-forward image of the virtual fundamental class of $\mathcal{M}_{e_{II;i}}$ into $\mathcal{A}.(\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C}))$ is equal to

$$c_{top}(\pi^*_{\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})}(\mathbf{W}_{II;i}\oplus\mathbf{C})\otimes\mathbf{H}_{II;i})\cap[\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})]$$
$$= c_{top}(\pi^*_{\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})}\mathbf{W}_{II;i}\otimes\mathbf{H}_{II;i})\cap c_1(\mathbf{H}_{II;i})\cap[\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})]$$

$$= c_{top}(\pi^*_{\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})}\mathbf{W}_{II;i}\otimes\mathbf{H}_{II;i})\cap[\mathbf{P}(\mathbf{V}_{II;i})],$$

[16] because $[\mathbf{P}(\mathbf{V}_{II;i})] = c_1(\mathbf{H}_{II;i})\cap[\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})]$.

We introduce the following short-hand notation which will be used frequently later.

Definition 8 *Define $\mathbf{U}_{e_{II;1},\cdots,e_{II;p}}$ to be*

$$\oplus_{1\leq i\leq p}\pi^*_{\times_B^{1\leq i\leq p}\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})}(\mathbf{V}_{II;i})\otimes\mathbf{H}_{II;i}.$$

The family moduli space $\mathcal{M}_{\underline{C}}$ is embedded in $\mathbf{P}(\mathbf{V}_{\underline{C}})$ as a projectified abelian cone and it is the zero locus $Z(s_{\underline{C}})$ of a section $s_{\underline{C}}$ of $\pi^*_{\mathbf{P}(\mathbf{V}_{\underline{C}})}\mathbf{W}_{\underline{C}}\otimes\mathbf{H}$. The algebraic family Seiberg-Witten invariant of $\underline{C}$ is defined to be the integral of the top intersection pairing of $c_{top}(\pi^*_{\mathbf{P}(\mathbf{V}_{\underline{C}})}\mathbf{W}_{\underline{C}}\otimes\mathbf{H})$ capping with a suitable power of $c_1(\mathbf{H})$.

We pull back the algebraic family Kuranishi model $(\Phi_{\mathbf{V}_{\underline{C}}\mathbf{W}_{\underline{C}}},\mathbf{V}_{\underline{C}},\mathbf{W}_{\underline{C}})$ from B to $\times_B^{1\leq i\leq p}\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})$. To simplify our notation, we skip the pull-back notation and denote the datum of the Kuranishi models by the same symbols.

By remark 8, we have to extend the obstruction bundle from $\pi^*_{\mathbf{P}(\mathbf{V}_{\underline{C}})}\mathbf{W}_{\underline{C}}\otimes\mathbf{H}$ to $\pi^*_{\mathbf{P}(\mathbf{V}_{\underline{C}})}\mathbf{W}_{\underline{C}}\otimes\mathbf{H}\oplus\mathbf{U}_{e_{II;1},\cdots,e_{II;p}}$, or equivalently to $\pi^*_{\mathbf{P}(\mathbf{V}_{\underline{C}})}\mathbf{W}_{\underline{C}}\otimes\mathbf{H}\otimes_{i\leq p}\mathbf{H}_{II;i}\oplus\mathbf{U}_{e_{II;1},e_{II;2},\cdots,e_{II;p}}$. And then insert $[B]_p$ into the intersection pairing.

Remark 9 *The above twisting of $\pi^*_{\mathbf{P}(\mathbf{V}_{\underline{C}})}\mathbf{W}_{\underline{C}}\otimes\mathbf{H}$ by $\otimes_{i\leq p}\mathbf{H}_{II;i}$ does not affect the family invariant because the embedding $\sigma:\mathcal{T}_B(\mathcal{X})\mapsto\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})$ defined by remark 8 is totally disjoint from the smooth divisor at infinity $\cong\mathbf{P}(\mathbf{V}_{II;i})$ and the line bundle $\mathbf{H}_{II;i}$ is trivialized over $\sigma(\mathcal{T}_B(\mathcal{X}))$.*

On the other hand, the moduli space of co-existence of $e_{II;1},\cdots,e_{II;p}$, $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$, is a closed sub-scheme of the auxiliary base space $B' = \times_B^{1\leq i\leq p}\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})$. So $\mathcal{M}_{\underline{C}}\times_{B'}\mathcal{M}_{e_{II;1},e_{II;2},\cdots,e_{II;p}}$ is also a closed sub-scheme of the projective bundle $X = \mathbf{P}(\mathbf{V}_{\underline{C}})$. By section II.7, page 160-161 of [Ha], we may blow up $X = \mathbf{P}_{B'}(\mathbf{V}_{\underline{C}})$ along $Z(s_{\underline{C}})\times_{B'}\mathcal{M}_{e_{II;1},\cdots,e_{II;p}} = \mathcal{M}_{\underline{C}}\times_{B'}\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ and make it into a divisor $\mathbf{D}$ of the blown up scheme $\tilde{X}$. And the direct application of the residual intersection formula implies that we may rewrite

$$c_{top}(\pi^*_{\tilde{X}}\mathbf{W}_{\underline{C}}\otimes\mathbf{H}\otimes_{j\leq p}\mathbf{H}_{II;j}) = c_{top}(\pi^*_{\tilde{X}}\mathbf{W}_{\underline{C}}\otimes\mathbf{H}\otimes_{j\leq p}\mathbf{H}_{II;j}\otimes\mathcal{O}(-\mathbf{D}))$$

[16] $\mathbf{P}(\mathbf{V}_{II;i})$ can be viewed as the compactifying divisor at the infinity of $\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})$.

$$+\sum_{1\leq i\leq rank_{\mathbf{C}}\mathbf{W}_{\underline{C}}}(-1)^{i-1}c_{rank_{\mathbf{C}}\mathbf{W}_{\underline{C}}-i}(\pi_{\tilde{X}}^{*}\mathbf{W}_{\underline{C}}\otimes\mathbf{H}\otimes_{j\leq p}\mathbf{H}_{II;j})\cap\mathbf{D}^{i-1}[\mathbf{D}].$$

And the push-forward of

$$\sum_{1\leq i\leq rank_{\mathbf{C}}\mathbf{W}_{\underline{C}}}(-1)^{i-1}c_{rank_{\mathbf{C}}\mathbf{W}_{\underline{C}}-i}(\pi_{\tilde{X}}^{*}\mathbf{W}_{\underline{C}}\otimes\mathbf{H}\otimes_{j\leq p}\mathbf{H}_{II;j}\oplus\mathbf{U}_{e_{II;1},\cdots,e_{II;p}})$$

$$\cap\mathbf{D}^{i-1}[\mathbf{D}]\cap[B]_p\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}B+p_g-q+\frac{\underline{C}^2-c_1(\mathbf{K}_{\mathcal{X}/B})\cdot\underline{C}}{2}}$$

is the localized contribution of the algebraic family Seiberg-Witten invariant along $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$.

It is vital to understand:

Question: Is the localized contribution of top Chern class the correct "invariant" associated to the collection of type II classes $e_{II;1}$, $\cdots$, $e_{II;p}$? In other words, is the localized contribution of top Chern classes constructed above invariant under deformations of the family $\mathcal{X}\mapsto B$ or of the Kuranishi models?

When the exceptional classes are of type I, it has been shown in [Liu5], [Liu6] that the localized contribution of top Chern class can be identified with certain mixed family invariant of $C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i$ and consequentially are known to be topological. On the other hand, the theory for type II curves is more delicate than their type I counterpart as the naively chosen localized contribution may not always be an invariant.

To understand what may go wrong, one may consider some deformation of the datum $\Phi_{\mathbf{V}_{II}\mathbf{W}_{II}}:\mathbf{V}_{II}\mapsto\mathbf{W}_{II}$. In order for it the correct choice, the localized contribution of top Chern class has to be independent to the deformations. Consider the idealistic situation that under the one parameter family of deformations, the family moduli space $\mathcal{M}_{e_{II}}$ of a single e_{II} (i.e. $p=1$) is deformed into the whole space $\mathbf{P}(\mathbf{V}_{e_{II}}\oplus\mathbf{C})$. After such a degeneration[17], the family moduli space of e_{II} is apparently not of the expected dimension. It is easy to see that the localized contribution of top Chern class along such a degenerated family moduli space is nothing but the whole localized top Chern class of $\mathbf{H}\otimes\pi_X^*\mathbf{W}_{\underline{C}}$ along $\mathcal{M}_{\underline{C}}$.

On the other hand, for the original well-behaved $\mathcal{M}_{e_{II}}$ the localized contribution of top Chern class is usually not equal to the whole localized top Chern class of $\mathbf{H}\otimes\pi_X^*\mathbf{W}_{\underline{C}}$ along $\mathcal{M}_{\underline{C}}$. Therefore we observe in this hypothetical example that the localized contribution of top Chern classes may **not** be invariant to the degenerations.

We also realize from this example that the non-invariant nature of the localized contribution of top Chern classes is due to the non-invariant nature of the family moduli space $\mathcal{M}_{e_{II}}$!

[17]Geometrically this corresponds to thickening the family moduli space of e_{II} to the whole space, which can be achieved by multiplying the map $\Phi_{\mathbf{V}_{e_{II}}\mathbf{W}_{e_{II}}}$ by a constant t and shrink t to zero.

Special Assumption: In the following, we assume that $\mathcal{M}_{\underline{C}-\sum e_{II;i}}\times_B \mathcal{M}_{e_{II;1},\cdots,e_{II;p}}\hookrightarrow \mathcal{M}_{\underline{C}}\times_B \mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ induced by adjoining the union of type II curves in $\sum_{i\leq p}e_{II;i}$ to a curve in $\underline{C}-\sum_{i\leq p}e_{II;i}$ has been an isomorphism.

This assumption is the analogue of the simplifying assumption of theorem 4. in [Liu5].

The following is the main theorem in this paper,

Theorem 1 *Given the localized contribution of top Chern class*

$$\sum_{1\leq i\leq rank_{\mathbf{C}}\mathbf{W}_{\underline{C}}}(-1)^{i-1}c_{rank_{\mathbf{C}}\mathbf{W}_{\underline{C}}-i}(\pi_{\tilde{X}}^{*}\mathbf{W}_{\underline{C}}\otimes \mathbf{H}\otimes_{j\leq p}\mathbf{H}_{II;j}\oplus \mathbf{U}_{e_{II;1},\cdots,e_{II;p}})$$

$$\cap \mathbf{D}^{i-1}[\mathbf{D}]\cap [B]_p\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}B+p_g-q+\frac{\underline{C}^2-c_1(\mathbf{K}_{\mathcal{X}/B})\cdot\underline{C}}{2}},$$

then under the above **special assumption**, *it can be expanded into an algebraic expression of cycle classes. Among the various terms of the expansion, there is a dominating term proportional to the virtual fundamental class* $[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}$, *and is identified to be*

$$\mathcal{AFSW}_{\mathcal{X}\mapsto B}((\times_B^{i\leq p}\pi_i)_*\big([\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}\cap[B]_p\big)\cap c_{p_g}^p(\mathcal{R}^2\pi_*\mathcal{O}_{\mathcal{X}})\cap\tau,\underline{C}-\sum_{1\leq i\leq p}e_{II;i}).$$

It satisfies the crucial invariant properties,

(i). The class $\tau\in\mathcal{A}.(\mathcal{M}_{e_{II;1},\cdots,e_{II;p}})$ *is independent to* nD *and is depending on* $\underline{C}$ *and* $e_{II;1},\cdots,e_{II;p}$ *only.*

(ii). When $febd(e_{II;i},\mathcal{X}/B)=p_g$, $1\leq i\leq p$, *and* $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ *is smooth of its expected dimension* $dim_{\mathbf{C}}B+p\cdot p_g+\sum\frac{e_{II;i}^2-c_1(\mathbf{K}_{\mathcal{X}}/B)\cdot e_{II;i}}{2}$, *the smooth cycle* $[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]$ *coincide with the virtual fundamental class* $[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}$ *and the above mixed invariant can be re-expressed as*

$$\mathcal{AFSW}_{\mathcal{X}\times_B\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}\mapsto\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}}([B]_p\cap\tau,\underline{C}-\sum_{1\leq i\leq p}e_{II;i}).$$

(iii). The above expression satisfies guidelines 1-3 listed beginning from page 20.

Proof of theorem 1: After we push-forward along $\mathbf{D}\mapsto\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$, the localized contribution of top Chern classes of $\pi_X^*\mathbf{W}_{\underline{C}}\otimes\mathbf{H}\otimes_{i\leq p}\mathbf{H}_{II;i}$ along $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ is equal to $\{c_{total}(\pi_X^*\mathbf{W}_{\underline{C}}\otimes\mathbf{H}\otimes_{i\leq p}\mathbf{H}_{II;i}\oplus\mathbf{U}_{e_{II;1},e_{II;2},\cdots,e_{II;p}})\cap s_{total}(\mathcal{M}_{\underline{C}}\times_B\mathcal{M}_{e_{II;1},\cdots,e_{II;p}},\mathbf{P}(\mathbf{V}_{\underline{C}})\times_B B'))\cap[B]_p\}_{dim_{\mathbf{C}}B+p_g-q+\frac{\underline{C}^2-\underline{C}\cdot c_1(\mathbf{K}_{\mathcal{X}/B})}{2}}$.

Assuming that $\mathcal{M}_{\underline{C}-\sum_{i\leq p}e_{II;i}}\times_B\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}=\mathcal{M}_{\underline{C}}\times_B\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$, the Segre class of the normal cone of $\mathcal{M}_{\underline{C}}\times_B\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ is the same as the Segre class of $\mathcal{M}_{\underline{C}-\sum_{i\leq p}e_{II;i}}\times_B\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}\subset\mathbf{P}(\mathbf{V}_{\underline{C}})$.

Step I: Recall the following short exact sequence[18] in subsection 4.1,

$$0 \mapsto \mathbf{W}_{\underline{C}-\sum_{i\leq p} e_{II;i}} \mapsto \mathbf{W}_{\underline{C}} \otimes_{i\leq p} \mathbf{H}_{II;i} \mapsto \mathbf{W}' \mapsto 0.$$

By applying lemma 5 and the discussion in subsection 4.0.1, we can view the above localized contribution of top Chern class as the Gysin pull-back $\Delta^!$ (with $\Delta : B \mapsto B \times B$) from $\mathcal{A}.(\mathcal{M}_{\underline{C}-\sum e_{II;i}} \times \mathcal{M}_{e_{II;1},\cdots,e_{II;p}})$. We may rewrite the original localized contribution of top Chern class as the $\Delta^!$ pull-back of

$$\{c_{total}(\pi_X^*\mathbf{W}_{\underline{C}-\sum e_{II;i}} \otimes \mathbf{H}) \cap s_{total}(\mathcal{M}_{\underline{C}-\sum_{i\leq p} e_{II;i}}, \mathbf{P}(\mathbf{V}_{\underline{C}-\sum e_{II;i}}))$$

$$\times c_{total}(\mathbf{W}' \otimes \mathbf{H}) \cap s_{total}(\mathcal{M}_{e_{II;1},e_{II;2},\cdots,e_{II;p}}, B') \cap c_{total}(\mathbf{U}_{e_{II;1},\cdots,e_{II;p}})$$
$$\cap s_{total}(\mathbf{V}' \otimes \mathbf{H}) \cap [B]_p\}_{2dim_{\mathbf{C}}B+p_g-q+\frac{\underline{C}^2-c_1(\mathbf{K}_{\mathcal{X}/B})\cdot\underline{C}}{2}}.$$

We have used the short exact [19] sequence $0 \mapsto \mathbf{V}_{\underline{C}-\sum e_{II;i}} \mapsto \mathbf{V}\otimes_{i\leq p}\mathbf{H}_{II;i} \mapsto \mathbf{V}' \mapsto 0$ over $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ and the following identity

$$s_{total}(\mathcal{M}_{\underline{C}-\sum e_{II;i}} \times_{B'} \mathcal{M}_{e_{II;1},\cdots,e_{II;p}}, \mathbf{P}(\mathbf{V}_{\underline{C}}))$$
$$= s_{total}(\mathcal{M}_{\underline{C}-\sum e_{II;i}} \times_{B'} \mathcal{M}_{e_{II;1},\cdots,e_{II;p}}, \mathbf{P}(\mathbf{V}_{\underline{C}-\sum e_{II;i}})) \cap s_{total}(\mathbf{V}' \otimes \mathbf{H}).$$

Based on the following identity

$$dim_{\mathbf{C}}B-q+p_g+\frac{(\underline{C}-\sum_{i\leq p} e_{II;i})^2 - c_1(\mathbf{K}_{\mathcal{X}/B})\cdot(\underline{C}-\sum_{i\leq p} e_{II;i})}{2}+\sum_{i\leq p}\frac{e_{II;i}^2 - c_1(\mathbf{K}_{\mathcal{X}/B})\cdot e_{II;i}}{2}$$

$$= dim_{\mathbf{C}}B-q+p_g+\frac{\underline{C}^2 - c_1(\mathbf{K}_{\mathcal{X}/B})\cdot\underline{C}}{2}+\{\sum_{i\leq p}(-\underline{C}\cdot e_{II;i}+e_{II;i}^2)+\sum_{1\leq i<j\leq p} e_{II;i}\cdot e_{II;j}\},$$

on the family dimensions, we may set:
$a_1 = dim_{\mathbf{C}}B - q + p_g + \frac{(\underline{C}-\sum e_{II;i})^2 - c_1(\mathbf{K}_{\mathcal{X}/B})\cdot(\underline{C}-\sum e_{II;i})}{2}$, $a_2 = dim_{\mathbf{C}}B + \sum\frac{e_{II;i}^2 - c_1(\mathbf{K}_{\mathcal{X}/B})\cdot e_{II;i}}{2}$ and $a_3 = \sum(-\underline{C}\cdot e_{II;i} + e_{II;i}^2) + \sum_{1\leq i<j\leq p} e_{II;i}\cdot e_{II;j}$ and focus on the term [20]

$$g_{a_1,a_2,a_3} = \{c_{total}(\pi_X^*\mathbf{W}_{\underline{C}-\sum e_{II;i}}\otimes\mathbf{H})\cap s_{total}(\mathcal{M}_{\underline{C}-\sum_{i\leq p} e_{II;i}}, \mathbf{P}(\mathbf{V}_{\underline{C}-\sum e_{II;i}}))\}_{a_1}$$

[18] Check the statement of proposition 6 for the definition of $\mathbf{W}'$.
[19] Check the commutative diagram in proposition 2 for the $p = 1$ version.
[20] Assume that a_3 satisfies $a_3 - p\cdot q(M) \geq 0$. Also notice that we have the identity $a_1 + a_2 - a_3 = 2dim_{\mathbf{C}}B - q(M) + p_g + \frac{\underline{C}^2-c_1(\mathbf{K}_{\mathcal{X}/B})\cdot\underline{C}}{2}$.

$\times\{c_{total}(\oplus_{i\leq p}\mathbf{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\big)\otimes\mathbf{H}_{II;i}\oplus\mathbf{W}'\otimes\mathbf{H})\cap s_{total}(\mathcal{M}_{e_{II;1},e_{II;2},\cdots,e_{II;p}},B')\cap_{i\leq p}c_1(\mathbf{H}_{II;i})\}_{a_2}$

$\cap\{c_{a_3-p\cdot q(M)}(\mathbf{U}_{e_{II;1},\cdots,e_{II;p}}-\oplus_{i\leq p}(\mathbf{C}\oplus\mathbf{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\big))\otimes\mathbf{H}_{II;i}-\mathbf{V}'\otimes\mathbf{H})\cap[B]_p\}.$

The reason to pick this particular combination will be clear momentarily.

(i). By using $\cap_{i\leq p}c_1(\mathbf{H}_{II;i})\cap[B'] = [\times_B^{i\leq p}\mathbf{P}(\mathbf{V}_{II;i})]$, the expression $\{c_{total}(\pi_X^*\mathbf{W}_{\underline{C}-\sum e_{II;i}}\otimes \mathbf{H})\cap s_{total}(\mathcal{M}_{\underline{C}-\sum_{i\leq p}e_{II;i}},\mathbf{P}(\mathbf{V}_{\underline{C}-\sum e_{II;i}}))\}_{a_1}$ is nothing but the virtual fundamental class of $\mathcal{M}_{\underline{C}-\sum e_{II;i}}$, expressed as the localized top Chern class of $\pi_X^*\mathbf{W}_{\underline{C}-\sum e_{II;i}}\otimes\mathbf{H}$.

(ii). By applying proposition 6 to [21] $\{c_{total}((\oplus_{i\leq p}\mathbf{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\big)\oplus\mathbf{C})\otimes\mathbf{H}_{II;i}\oplus \mathbf{W}'\otimes\mathbf{H})\cap s_{total}(\mathcal{M}_{e_{II;1},e_{II;2},\cdots,e_{II;p}},B')\}_{a_2}$, it can be expanded into an algebraic expression of cycle classes and the leading term is $[\mathcal{M}_{e_{II;1},e_{II;2},\cdots,e_{II;p}}]_{vir}\cap c_{p_g}(\mathcal{R}^2\pi_*\mathcal{O}_{\mathcal{X}})^p$.

Step II: Because both $[\mathcal{M}_{\underline{C}-\sum e_{II;i}}]_{vir}$ and $[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}$ are well defined and are independent to nD, for the whole expression to be nD independent we evaluate the nD independent term of

$$\{c_{a_3-p\cdot q(M)}(\mathbf{U}_{e_{II;1},\cdots,e_{II;p}}-\oplus_{i\leq p}(\mathbf{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\big)\oplus\mathbf{C})\otimes\mathbf{H}_{II;i}-\mathbf{V}'\otimes\mathbf{H})\cap[B]_p\}.$$

Consider the virtual bundle $\omega = \mathbf{U}_{e_{II;1},e_{II;2},\cdots,e_{II;p}} - \mathbf{V}'\otimes\mathbf{H} - \oplus_{i\leq p}(\mathbf{C}\oplus \mathbf{R}^0\pi_*\big(\mathcal{O}_{nD}(nD)\big))\otimes\mathbf{H}_{II;i}$.

The virtual bundle ω is of virtual rank $\sum_{i\leq p}\chi(\mathcal{O}_{\mathcal{X}_b}(e_{II;i}+nD))-\chi(\mathcal{O}_{\sum \mathbf{e}_{II;i}|_b}(\underline{C}+nD)) - p(\chi(\mathcal{O}_{\mathcal{X}_b}(nD))+q)$ ($\mathcal{X}_b$ is the fiber of a closed $b\in B$) and by surface Riemann-Roch formula it is equal to

$$rank(\omega) = -q(M)\cdot p+\sum_{i\leq p}(e_{II;i})^2-\underline{C}\cdot(\sum_{i\leq p}e_{II;i})+\sum_{i<j\leq p}e_{II;i}\cdot e_{II;j} = a_3-p\cdot q(M).$$

The expression $rank(\omega) = a_3-(dim_{\mathbf{C}}\times_B^p\mathcal{T}_B\mathcal{X}-dim_{\mathbf{C}}B)$ is nD independent!

Definition 9 *Consider the nD independent term of $c_{rank(\omega)}(\omega)$ and expand it as a polynomial of $c_1(\mathbf{H})$, $\sum_r \tau_r c_1(\mathbf{H})^r$. Define the τ class to be the sum of τ_r, $\tau = \sum_{r\leq rank(\omega)}\tau_r$.*

If we view $c_1(\mathbf{H})$ as a formal variable z, then τ can be viewed as $c_{rank(\omega)}(\omega)|_{z=1}^{n=0}$.

Recall (e.g. chapter 15, page 281 of [F]) that for a proper morphism f and a coherent sheaf $\mathcal{F}$, $f_*\mathcal{F} = \sum_{i\geq 0}(-1)^i\mathcal{R}^if_*\mathcal{F}$.

The following lemma identifies the nD independent term of $c_{rank(\omega)}(\omega)$.

[21] Notice that we have added $\oplus_{i\leq p}\mathbf{C}\otimes\mathbf{H}_{II;i}$ to our bundle here. It is to match with the additional $\mathbf{H}_{II;i}$ factor in the stabilized obstruction bundle of $e_{II;i}$, $(\pi^*_{\mathbf{P}(\mathbf{V}_{II;i}\oplus\mathbf{C})}\mathbf{W}_{II;i}\oplus \mathbf{C})\otimes\mathbf{H}_{II;i}$.

Lemma 8 *The nD independent term of $c_{rank(\omega)}(\omega)$ is equal to $c_{rank(\omega)}(\oplus_{i\leq p}\pi_*\mathcal{O}(\mathbf{e}_{II;i})-\pi_*\mathcal{O}_{\sum_{i\leq p}\mathbf{e}_{II;i}}(\underline{C})\otimes\mathbf{H})$.*

Proof of lemma 8: When D is very ample and $n\gg 0$, the Serre vanishing implies $\mathcal{R}^0\pi_*\mathcal{O}_{\sum\mathbf{e}_{II;i}}(\underline{C}+nD)=\pi_*\mathcal{O}_{\sum\mathbf{e}_{II;i}}(\underline{C}+nD)$, etc. This enables us to re-express ω as the differences of several direct images. Finally we set $n=0$ in the alternative expression of ω. $\Box$

Now we may express the nD independent leading term of g_{a_1,a_2,a_3} as

$$[\mathcal{M}_{\underline{C}-\sum_{i\leq p}e_{II;i}}]_{vir}\times\{[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}\cap c^p_{p_g}(\mathcal{R}^2\pi_*\mathcal{O}_{\mathcal{X}})\}\cap(\sum_r\tau_r c_1(\mathbf{H})^r).$$

So the dominating term of the original localized top Chern class becomes

$$\sum_{r\leq rank(\omega)}\Delta^!\{[\mathcal{M}_{\underline{C}-\sum_{i\leq p}e_{II;i}}]_{vir}\cap c_1(\mathbf{H})^r\times[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}\cap c^p_{p_g}(\mathbf{R}^2\pi_*\mathcal{O}_{\mathcal{X}})\cap[B]_p\cap\tau_r\},$$

which is nothing but

$$\sum_r\{[\mathcal{M}_{\underline{C}-\sum_{i\leq p}e_{II;i}}]_{vir}\cap c_1(\mathbf{H})^r\}\cdot_{id_B}\{[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}\cap c^p_{p_g}(\mathbf{R}^2\pi_*\mathcal{O}_{\mathcal{X}})\cap[B]_p\cap\tau_r\}.$$

After we cap with $c_1(\mathbf{H})^{dim_{\mathbf{C}}B-q+p_g+\frac{\underline{C}^2-c_1(\mathbf{K}_{\mathcal{X}/B})\cap\underline{C}}{2}}$ and push forward the top intersection pairing to a point pt, the top intersection pairing is reduced to $\mathcal{AFSW}_{\mathcal{X}\mapsto B}((\times_B^{i\leq p}\pi_i)*\{[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}\cap[B]_p\cap\tau\}\cap c^p_{p_g}(\mathbf{R}^2\pi_*\mathcal{O}_{\mathcal{X}}),\underline{C}-\sum_{i\leq p}e_{II;i})$.

By our computation, it clearly obeys guideline 1 on page 20.

To show that it is compatible with the type I theory in [Liu5], [Liu6], we notice that $[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}$ is reduced to $\cap_{i\leq p}[Y(\Gamma_{e_{k_i}})]=[Y(\Gamma)]$ when the type II classes are replaced by e_{k_i}, $1\leq i\leq p$. On the other hand, we may view $c_1(\mathbf{H})$ as a formal variable z, then formally $\tau=c_{rank(\omega)}(\omega)|^{n=0}_{z=1}$. When the type II classes are reduced to the type I classes $e_{k_1},e_{k_2},\cdots,e_{k_p}$, the argument of theorem 4 of [Liu5] or proposition 18 of [Liu6] implies that the effect of $c_{rank(\omega)}(\omega)\cap[B]_p|_{n=0}$ upon the fundamental class $[Y(\Gamma)]$ is equal to $c_{rank(\omega)+p\cdot q(M)}(\mathcal{R}^1\pi_*\mathcal{O}_{\sum\mathbf{e}_{k_i}}(\underline{C})\otimes\mathbf{H}-\oplus_i\mathcal{R}^1\pi_*\mathcal{O}_{\mathbf{e}_{k_i}}(\mathbf{e}_{k_i}))$ and it is equivalent to the top Chern class of an explicit vector bundle representative [22] of $\tau_\Gamma\otimes\mathbf{H}$. So $\tau=c_{top}(\tau_\Gamma\otimes\mathbf{H})|^{n=0}_{z=1}=c_{total}(\tau_\Gamma)$.

The only defect between the degenerated version of type II theory and the original type I theory is the expression $\cap c^p_{p_g}(\mathcal{R}^2\pi_*\mathcal{O}_{\mathcal{X}})$. This defect roots at the discrepancy of their dimension formulae and should be discarded when the classes are of type I. With this defect removed, the type II contribution for

[22] By lemma 17 of [Liu6].

$\underline{C} - \sum e_{II;i}$ is reduced to $\mathcal{AFSW}_{M_{n+1}\times Y(\Gamma)\mapsto M_n\times Y(\Gamma)}(1, C - \mathbf{M}(E)E - \sum e_{k_i})$ when we take $\underline{C} = C - \mathbf{M}(E)E$.

The object we have identified is independent to nD because of the $nD-$independence of $[\mathcal{M}_{\underline{C}-\sum e_{II;i}}]_{vir}$ and $[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}$ and the $nD-$independence of τ.

Thus our construction obeys guidelines 1-3 starting from page 20, and the theorem is proved. $\Box$

Remark 10 *When $febd(e_{II;i}, \mathcal{X}/B) = 0$ for all $e_{II;i}$, the factor $c^p_{p_g}(\mathcal{R}^2\pi_*\mathcal{O}_{\mathcal{X}})$ disappears from the mixed invariant. The mixed invariant identified in the theorem should be replaced by*

$$\mathcal{AFSW}_{\mathcal{X}\mapsto B}((\times_B^{i\leq p}\pi_i)_*\big([\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}\cap[B]_p\big)\cap\tau, \underline{C} - \sum_{1\leq i\leq p} e_{II;i}).$$

Remark 11 *From the above discussion at the end of our main theorem, the τ class defined above is the type II analogue of $c_{total}(\tau_\Gamma)$ defined for type I theory. Its role is to balance the rank difference between the family obstruction bundles of $\underline{C}$ and of $\underline{C} - \sum_{i\leq p} e_{II;i}$. The main difference from the type I theory is that for τ_Γ we can represent it as a vector bundle and $c_{top+l}(\tau_\Gamma) = 0$ for all $l > 0$. But ω is only a virtual vector bundle of virtual rank $-p\cdot q(M) - \sum_{i\leq p}\underline{C}\cdot e_{II;i} + \sum_{i\leq p} e^2_{II;i} + \sum_{i<j} e_{II;i}\cdot e_{II;j}$. As the type II universal curves $\mathbf{e}_{II;i}$ may behave badly (in the sense described on page 4), we do not expect to find an explicit bundle representative of ω generally.*

4.3 The Remark about the Inductive Scheme of Applying Residual Intersection Formula

At the end of the whole paper, we sketch the extension of the theory to an inductive scheme upon a whole hierarchy of collection of type II curves and explain how it fits to our guideline 4 on page 20. As the current discussion is parallel to type I theory in [Liu6], we do not intend to go into the full details here. The reader who wants to get to the detailed arguments can consult [Liu6]. By combining with the main theorem in the current paper, the reader can translate the argument to cover the type II case.

There are a few reasons why the theory of type II curves has to be extended beyond a single collection of type II classes.

(1). Exactly parallel to the type I classes, type II curves can break, or degenerate into a union of irreducible curves, while some of them are again type II curves. The degenerated configuration will give additional excess contributions.

(2). The main theorem in the paper has been proved under the **special assumption** that

$$\mathcal{M}_{\underline{C}-\sum e_{II;i}}\times_{B'}\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}\hookrightarrow\mathcal{M}_{\underline{C}}\times_{B'}\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$$

is isomorphic. In section 4.2 this has been interpreted equivalently as the injectivity of the bundle map $\pi^*_{\mathbf{P}(\mathbf{V}_{\underline{C}})}\mathbf{W}_{new}\otimes\mathbf{H} \mapsto \pi^*_{\mathbf{P}(\mathbf{V}_{\underline{C}})}\mathbf{W}_{\underline{C}}\otimes\mathbf{H}$ over $\mathcal{M}_{\underline{C}-\sum e_{II;i}} \times_{B'} \mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$.

In general, the breaking up of the type II classes into irreducible components may cause the inclusion failing to be isomorphic. In other words, a curve in $\underline{C}$ above the locus of co-existence $\cap_{i\leq p}\pi_i\mathcal{M}_{e_{II;i}}$ may not factorize into a curve in $\underline{C} - \sum e_{II;i}$ and curves in the sum of $\sum_{i\leq p} e_{II;i}$ when some curve in $e_{II;i}$ fails to be irreducible.

Both theories of type I and type II classes make use of residual intersection theory of top Chern classes. The major difference between type II theory and their type I counter-part is that the localized contributions of top Chern classes of type I classes are identifiable to be mixed family invariants, while the localized contribution of top Chern classes of type II exceptional classes are typically non-topological. One major issue we have pointed out is that $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ may not always be regular and the excess invariant contribution within $\mathcal{M}_{\underline{C}} \times_{B'} \mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ may depend on the explicit locus $\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ rather than the virtual fundamental class $[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}$.

Our main theorem on page 28 demonstrates that despite the non-topological nature of the localized contribution of top Chern class, it can still be expanded algebraically and the dominating term is of the desired form:

$$\mathcal{AFSW}_{\mathcal{X}\mapsto B}((\times_B^{i\leq p}\pi_i)_*([\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir})\cap c^p_{p_g}(\mathcal{R}^2\pi_*\mathcal{O}_{\mathcal{X}})\cap[B]_p\cap\tau, \underline{C}-\sum_{i\leq p} e_{II;i}).$$

This corresponds to the invariant contribution proportional to $[\mathcal{M}_{\underline{C}-\sum_{i\leq p} e_{II;i}}]_{vir}\cdot_{id_B} [\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}$. In this way, the non-topological nature of the localized contribution of top Chern classes is from the other non-dominating terms from $\mathcal{M}_{\underline{C}} \times_{B'} \mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ away from an explicit cycle representative of $[\mathcal{M}_{\underline{C}}]_{vir}\cdot_{id_B} [\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}$. Unless we impose additional assumptions, we do not expect any vanishing result on these correction terms.

By re-grouping the correction terms with the residual contribution of top Chern class

$$\int_{\tilde{X}} c_{top}(\pi^*_{\tilde{X}}\mathbf{W}_{\underline{C}}\otimes\mathbf{H}\otimes\mathcal{O}(-\mathbf{D})\oplus\mathbf{U}_{e_{II;1},\cdots,e_{II;p}})\cap[B]_p\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}B-q+p_g+\frac{\underline{C}^2-c_1(\mathbf{K}_{\mathcal{X}/B})\cdot\underline{C}}{2}},$$

their total sum is still a topological invariant!

This interpretation allows us to formulate our scheme inductively.

(i). List all the possible finite collections of type I and type II classes satisfying: $\underline{C}\cdot e_{k_i} < 0$, $e_{k_i}\cdot e_{k_j} \geq 0$, $i\neq j$, $\underline{C}\cdot e_{II;i} < 0$, $e_{II;i}\cdot e_{II;j}\geq 0$, $i\neq j$.

These exceptional classes determine exceptional cones in the sense of [Liu4].

(ii). Define a partial ordering among all such collections based on the inclusions of exceptional cones. The partial ordering encodes whether one particular

collection of exceptional classes degenerates into another. In the type I theory, such a partial ordering has been named $\succ$. Check definition 8 of [Liu6] for details.

(iii). Based on the partial ordering, define a linear ordering among the various collections of exceptional classes. This is the analogue of $\models$ defined in [Liu6].

(iv). Blow up $\mathbf{P}(\mathbf{V}_{\underline{C}})$ along the various sub-loci $\mathcal{M}_{\underline{C}} \times_{B'} \mathcal{M}_{e_{k_1},\cdots,e_{k_p};e_{II;1},\cdots,e_{II;q}}$ by the reversed linear ordering constructed in (iii). Each time we have to stabilize the family Kuranishi model of $\underline{C}$ following the recipe on page 25.

(v). From each localized contribution of top Chern class along $\mathcal{M}_{\underline{C}} \times_{B'} \mathcal{M}_{e_{k_1},\cdots,e_{k_p};e_{II;1},\cdots,e_{II;q}}$, we need to go through the computation in theorem 1 and identify its dominating term with the appropriated mixed family invariant. However, there are two subtle variations here.

(v'). The first variation from our main theorem is that under an inductive blowing up procedure, the obstruction bundle $\pi_X^* \mathbf{W}_{\underline{C}} \otimes \mathbf{H}$ has been modified repeatedly.

As in the type I case, we use proposition 9 of [Liu6] to deal with this question. The cited proposition demonstrates that the blowing ups performed ahead of the given one (under the reversed linear ordering in (iii).) changes the obstruction bundle in a way which enables us to relate the modified bundle with the bundle $\pi^*_{\mathbf{P}(\mathbf{V}_{\underline{C}})} \mathbf{W}_{\underline{C}-\sum e_{II;i}} \otimes \mathbf{H}$, exactly allowing us to drop the **special assumption** on page 28.

A special partial ordering parallel to $\sqsupset$ in definition 15 of [Liu6] has to be used to analyze the discrepancy between $\mathcal{M}_{\underline{C}-\sum e_{II;i}} \times_B \mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$ from $\mathcal{M}_{\underline{C}} \times_B \mathcal{M}_{e_{II;1},\cdots,e_{II;p}}$.

(v"). To avoid over-counting, the inductive blowup process (compare with section 5.1. of [Liu6]) incorporates the inclusion-exclusion principle (see page 19-20 of [Liu7] for an elementary explanation) and we identify the dominating terms of the localized contributions to be "modified" family invariants.

For type I classes, their corresponding modified family invariants have been defined inductively based on a partial ordering $\gg$ (definition 11.) in [Liu6]. Please consult definition 13, 14 of the same paper for the definitions of the type I modified family invariants. For the combinations of type I and type II classes, we can generalize $\gg$ and extend the definition of modified family invariant accordingly.

At the end of the inductive procedure, we end up with a modified family invariant $\mathcal{AFSW}^*_{\mathcal{X}\mapsto B}(1, \underline{C})$. It is the subtractions of all the modified family invariants attached to the various collections of type I/II exceptional curves from the original family invariant $\mathcal{AFSW}_{\mathcal{X}\mapsto B}(1, \underline{C})$.

This modified family invariant resembles the virtual number count of smooth curves in $\underline{C}$ within the given family $\mathcal{X} \mapsto B$.

The above procedure is parallel to the theory of type I exceptional curves.

If we apply the above scheme to the type I and type II exceptional classes of the universal families, we can derive the following result,

Theorem 2 *Given an algebraic surface M and a line bundle $L \mapsto M$. For $\delta \leq \frac{c_1^2(L)+c_1(\mathbf{K}_M)\cdot c_1(L)}{2} + 1$, the "virtual number of δ-node nodal curves" in a generic δ dimensional linear system of $|L|$ is defined.*

Notice that $\delta!\times$ the above virtual number is interpreted as the equivalence of smooth curves within the universal family $M_{\delta+1} \mapsto M_\delta$, represented by some modified family invariant. Unlike the universality theorem which works for $5\delta-1$ very ample L, the above result does not guarantee the deformation invariance of these virtual numbers of nodal curves.

4.3.1 The Vanishing Result for $K3$ or T^4 and Yau-Zaslow Formula

When we apply the residual intersection theory of type II curves to the universal families of $K3$ or T^4, we get a surprising vanishing result discussed briefly in [Liu7].

Recall that the universality theorem asserts that for a general algebraic surface M and a $5\delta-1$ very ample L, the δ-node nodal curves in a generic δ dimensional sub-linear system of $|L|$ can be expressed in a degree δ universal polynomial of $c_1^2(K_M), c_1(K_M)\cdot c_1(L)$, $c_1(L)^2$, $c_2(M)$.

When we consider $M = K3$ (or T^4), K_M is trivial and the universal polynomial is reduced to a polynomial of $c_1(L)^2$ and $c_2(M)$. On the other hand, for rational nodal curves the self-intersection number $c_1(L)^2$ is constrained by the adjunction formula by $c_1(L)^2 = 2\delta - 2$.

So the universal polynomial has been reduced to a degree δ polynomial of $c_2(M)$ above.

On the other hand, the well known Yau-Zaslow formula [YZ] asserts that when we we put all the numbers of δ-node ($\delta \in \mathbf{N}$) nodal curves into a generating function, it can be identified with the power series

$$1 + \sum_{\delta\in\mathbf{N}} n_\delta q^\delta = \{\frac{1}{\prod_{i\geq 0}(1-q^i)}\}^{c_2(M)}.$$

The following vanishing result implies that the type II exceptional curves within the universal family contribute nothing to the family invariant $\mathcal{AFSW}_{M_{n+1}\times\{t_L\}\mapsto M_n\times\{t_L\}}(1, c_1(L) - \sum 2E_i)$.

Theorem 3 *The virtual number of δ-node nodal curves in a linear sub-system of $|L|$ on an algebraic $K3$ surface is equal to $\frac{1}{\delta!}\mathcal{AFSW}^*_{M_{n+1}\times\{t_L\}\mapsto M_n\times\{t_L\}}(1, c_1(L) - 2\sum_{i\leq\delta} E_i)$, the normalized type I modified algebraic family Seiberg-Witten invariant defined in section 5.2., remark 12 of [Liu6].*

Sketch of the Proof of theorem 3: We apply the above scheme to both type I and type II exceptional classes on the universal families $M_{n+1} \mapsto M_n$. So the $\delta!\times$ the virtual number of $\delta-$node nodal curves in a linear sub-system of $|L|$ is equal to

$\mathcal{AFSW}_{M_{n+1}\times\{t_L\}\mapsto M_n\times\{t_L\}}(1, c_1(L)-\sum 2E_i)$-correction terms from both type I and type II exceptional classes. Each correction term is a modified invariant attached to $[\mathcal{M}_{e_{k_1},e_{k_2},\cdots,e_{k_p};e_{II;1},\cdots,e_{II;p'}}]_{vir}$.

On the other hand, we may distinguish the collections of exceptional classes into two subsets. The first subset collects all the collections of type I exceptional classes, the second subset collects all the collections not entirely of type I exceptional classes. I.e. it contains all the collections consisting of either type II exceptional classes or consisting of both type I and type II exceptional classes.

Independent to the details of the τ class and $Y(\Gamma)$ or $[\mathcal{M}_{e_{II;1},\cdots,e_{II;p}}]_{vir}$, by theorem 1 the important characteristic of the mixed invariant involving one or more type II curve is that there is an additional insertion[23] of $c_1(\mathcal{R}^2\pi_*\mathcal{O}_{M_{\delta+1}})$ to the family invariant.

On the other hand, there is the following commutative diagram,

$$\begin{array}{ccc} M_{\delta+1} & \mapsto & M_\delta \\ \downarrow & & \downarrow \\ M\times M_\delta & \mapsto & M_\delta \end{array}$$

and the push-forward of $M_{\delta+1}\mapsto M_\delta$ factors through $M\times M_\delta\mapsto M_\delta$. So $\mathcal{R}^2\pi_*\mathcal{O}_{M_{\delta+1}}$ can be identified with $\mathcal{O}_{M_\delta}\otimes H^2(M,\mathcal{O}_M)$. In particular, this implies that $c_1(\mathcal{R}^2\pi_*\mathcal{O}_{M_{\delta+1}})=0$. This implies that all the mixed family invariants involving one or more type II classes are identically zero.

As all the modified invariants are defined inductively by the differences of the mixed invariants, all the modified family invariants involving type II exceptional curves vanish on the universal families of $K3$. Therefore all the correction terms are from the type I exceptional curves. So $\delta!\times$ the virtual number of nodal curves collapses to the type I modified family invariant

$$\mathcal{AFSW}^*_{M_{n+1}\times\{t_L\}\mapsto M_n\times\{t_L\}}(1, c_1(L)-2\sum_{i\le\delta}E_i).$$

The theorem is proved. $\Box$

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